OLD AND NEW PROBLEMS AND RESULTS IN COMBINATORIAL NUMBER THEORY

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1. INTRODUCTION

In the present work we will discuss various problems in elementary number theory, most of which have a combinatorial flavor. In general we will avoid classical problems, just mentioning references for the interested reader. We will almost never give proofs but on the other hand we will try to give as exact references as we can. We will restrict ourselves mostly to problems on which we worked for two reasons: (i) In order not to make the paper too long; (ii) We may know more about them than the reader.

Both the difficulty and importance of the problems discussed are very variable—some are only exercises while others are very difficult or even hopeless and may have important consequences or their eventual solution may lead to important advances and the discovery of new methods. Some of the problems we think are difficult may turn out to be trivial after all —this has certainly happened before in the history of the world with anyone who tried to predict the future. Here is an amusing case. Hilbert lectured in the early 1920's on problems in mathematics and said something like this—probably all of us will see the proof of the Riemann hypothesis, some of us (but probably not I) will see the proof of Fermat's last theorem, but none of us will see the proof that $2^{\sqrt{2}}$ is transcendental. In the audience was Siegel, whose deep research contributed decisively to the proof by Kusmin a few years later of the transcendence of $2^{\sqrt{2}}$. In fact shortly thereafter Gelfond and a few weeks later Schneider independently proved that α^β is transcendental if α and β are algebraic, β is irrational and $\alpha \neq 0, 1$. Thus, we hope the reader will forgive us if some (not many, we hope) of the problems turn out to be disappointingly simple.

Before starting, we mention a number of papers which also deal mainly with unsolved problems in combinatorial number theory. These references, which will not be included in the references at the end of the paper, will have an asterisk appended to them, for ease of later location.

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2. VAN DER WAERDEN'S THEOREM AND RELATED TOPICS

Denote by $W(n)$ the smallest integer such that if the (positive) integers not exceeding $W(n)$ are partitioned arbitrarily into two classes, at least one class always contains an arithmetic progression (A.P.) of length n. The celebrated theorem of van der Waerden [Wa (27)], [Wa (71)], [Gr-Ro (74)] shows that $W(n)$ exists for all n but all known proofs yield upper bounds on $W(n)$ which are extremely weak, e.g., they are not even primitive recursive functions of n. In the other direction, the best lower bound currently available (due to Berlekamp [Ber (68)]) is

$$W(n+1) > n \cdot 2^n$$

for n prime. It would be very desirable to know the truth here. The only values of $W(n)$ known (see [Chv (69)], [St-Sh (78)]) at present are:

$$W(2) = 3\,,\ W(3) = 9\,,\ W(4) = 35\,,\ W(5) = 178\,.$$

Recent results of Paris and Harrington [Par-Har (77)] show that certain combinatorial problems with a somewhat similar flavor (in particular, being variations of Ramsey's Theorem [Ramsey (30)], [Gr-Ro (71)]) do in fact have lower bounds which grow faster than any function which is provably recursive in first-order Peano arithmetic.

More than 40 years ago, Erdös and Turán [Er-Tu (36)], for the purposes of improving the estimates for $W(n)$, introduced the quantity $r_k(n)$, defined to be the least integer r so that if $1 \leqslant a_1 < \ldots < a_r \leqslant n$, then the sequence of a_i's must contain a k-term A.P. The best current bounds [Beh (46)], [Roth (53)], [Mo (53)] on $r_3(n)$ are

$$\frac{n}{\exp\left(c_1 \sqrt{\log n}\right)} < r_3(n) < \frac{c_2 n}{\log \log n}$$

where $c, c_1, c_2, \ldots$ will always denote suitable positive constants. Rankin [Ran (60)] has slightly better bounds for $r_k(n)$, $k > 3$. However, a recent stunning achievement of Szemerédi [Sz (75)] is the proof of the upper bound

$$r_k(n) = o(n).$$

His proof, which uses van der Waerden's theorem, does not give any usable bounds for $W(n)$. This result has also been proved in a rather different way by Furstenberg [Fu (77)] using ergodic theory. This proof also furnishes no estimate for $r_k(n)$. A much shorter version has recently been given by Katznelson and Ornstein (see [Tho (78)]). Perhaps

$$r_k(n) \stackrel{?}{=} o\left(\frac{n}{(\log n)^t}\right)$$

for every t. This would imply as a corollary that for every k there are k primes which form an A.P. The longest A.P. of primes presently known [Weint (77)] has length 17. It is $3430751869 + 87297210t$, $0 \leqslant t \leqslant 16$.

We next mention several conjectures which seem quite deep. They each would imply Szemerédi's theorem, for example.

The first one [1]) is this: Is it true that if a set A of positive integers satisfies $\sum_{a \in A} \frac{1}{a} = \infty$ then A must contain arbitrarily long A.P.'s?

Set

$$\alpha_k = \sup_{A_k} \sum_{a \in A_k} \frac{1}{a}$$

[1]) One of the authors (P.E.) currently offers US \$3000 for the resolution of this problem.

where A_k ranges over all sets of positive integers which do not contain a k-term A.P. As far as we know, $\alpha_k = \infty$ is possible, but this seems unlikely. The best lower bound known for α_k is due to Gerver [Ge (77)]:

$$\alpha_k \geqslant \bigl(1+o(1)\bigr)\, k \log k \, .$$

Trivially,

$$\alpha_k \geqslant \frac{1}{2} \log W(k) \, .$$

It would be interesting to show that

$$\alpha_k / \log W(k) > \frac{1}{2} + c$$

or even

$$\lim_{k \to \infty} \alpha_k / \log W(k) \to \infty$$

but at present we have no idea how to attack these questions.

The second conjecture is based on the following ideas. For a finite set $X = \{ x_1, ..., x_t \}$, let X^N denote the set of N-tuples $\{ (y_1, ..., y_N) : y_i \in X, 1 \leqslant i \leqslant N \}$. Call a set $P = \{ \bar{p}_1, \bar{p}_2, ..., \bar{p}_t \}$ of t N-tuples $\bar{p}_i \in X^N$ a *line* if the $\bar{p}_i$ have the following property: For each j, $1 \leqslant j \leqslant N$, *either* the jth component of $\bar{p}_i$ is x_i, $1 \leqslant i \leqslant t$, *or* all the jth components of the $\bar{p}_i$ are equal. Since $| P | = t$ then at least one j must satisfy the first condition. It is a theorem of Hales and Jewett [Hale-Je (63)] that for any r, if $N \geqslant N(t, r)$ then for any partition of X^N into r classes, some class must contain a line. This immediately implies van der Waerden's theorem by taking $x_i = i-1$, $1 \leqslant i \leqslant t$, and letting the N-tuple $(y_1, ..., y_N)$ correspond to the base t expansion of the integer $\sum_{i=1}^{N} y_i t^{i-1}$. In fact, it also implies the higher-dimensional generalizations of van der Waerden's theorem we shall mention shortly. The question now is this: Does the corresponding "density" result hold? In other words, is it true that for each $\varepsilon > 0$ and each integer t, there is an $N(t, \varepsilon)$ so that if $N \geqslant N(t, \varepsilon)$ and R is any subset of X^N satisfying $| R | > \varepsilon t^N$ then R contains a line P? (See also [Mo (70)], [Chv (72)]). For $t = 2$ each line P can be naturally associated with a pair of subsets $A, B \subseteq X$ with $A \subset B$. The truth of the conjecture for $t = 2$ then follows from the theorem of Sperner [Sper (28)] on the maximum size of a family of incomparable subsets of an N-set, namely, that such a family can have at most $\binom{N}{\left\lceil \frac{N}{2} \right\rceil} = o(2^N)$ sets. However for $t \geqslant 3$ the question is still wide open. Some recent partial results have been given by Brown [Bro (75)]. It is not even known whether for every c, $ck^N/\sqrt{N}$ points can be chosen without containing a line. (Also, see *Added in proof* p. 107.)

It is natural to ask whether analogues of van der Waerden's theorem hold in higher dimensions, i.e., for any finite subset S of the lattice points of $\mathbf{E}^n$ and any partition of the lattice points of $\mathbf{E}^n$ into two classes, at least one class contains a subset similar to S. That this is indeed the case was first shown by Gallai (see [Rad (33) b]) and independently by Witt [Wit (52)] and by Garsia [Gar (∞)]. The corresponding "density" results, i.e., the analogues of Szemerédi's theorem in higher dimensions, have very recently been proved by Furstenberg and Katznelson [Fu-Ka (78)] using techniques from ergodic theory. These would also follow from the truth of the "line" conjecture previously mentioned. It was previously shown by Szemerédi [Sz (∞)] (using $r_k(n) = o(n)$) that if R is a subset of $\{ (i,j): 1 \leqslant i, j \leqslant n \}$ with $| R | \geqslant \varepsilon n^2$ and $n \geqslant n(\varepsilon)$ then R must contain 4 points which form a square. Prior to that, Ajtai and Szemerédi [Aj-Sz (74)] had proved the analogous weaker result for the isoceles right triangle.

A recently very active area deals with various generalizations of the old result of Schur [Schur (16)] : For any partition of $\{ 1, 2, ..., [r!\, e] \}$ into r classes, the equation $x + y = z$ has a solution entirely in one class. This was generalized (independently) by Rado [Rad (70)], Sanders [San (68)], and Folkman (see [Gr-Ro (71)]) who showed that for any partition of $\mathbf{N}$ into finitely many classes, some class C must contain arbitrarily large sets $\{ x_1, x_2, ..., x_k \}$ such that all sums $\sum_{i=1}^{k} \varepsilon_i x_i$, $\varepsilon_i = 0$ or 1, belong to C. However, these results were subsumed by a fundamental result of Hindman [Hi (74)] who showed that under the same hypothesis, some class C must contain an *infinite* set $\{ x_1, x_2, ... \}$ such that all finite sums $\sum_{i=1}^{\infty} \varepsilon_i x_i$, $\varepsilon_i = 0$ or 1, belong to C (answering a conjecture of Graham and Rothschild and Sanders). Subsequently, simpler proofs were given by Baumgartner [Bau (74)] and Glazer [Gl (xx)]. Of course, the analogous result also holds for products (by restricting our attention to numbers of the form 2^x). A natural question to ask (see [Er (76) c*]) is whether some C must simultaneously contain infinite sets A and B such that all finite sums from A and all finite products from B are in C? Even more, is it possible that we could take $A = B$? In [Hi (79) b], [Hi (80)] Hindman shows that the answer to the first question is yes and the answer to the second question is no. In fact, he constructs a partition of N into two classes such that no infinite set $\{ x_1, x_2, ... \}$ has all its finite products and *pair* sums $x_i + x_j$, $i \neq j$, in one class. He also constructs a partition of **N** into *seven* classes so that no infinite set $\{ x_1, x_2, ... \}$ has all its pair products $x_i x_j$ and pair sums $x_i + x_j$, $i \neq j$, belonging to a single class. Whether arbitrarily large *finite* sets $\{ x_1, ..., x_k \}$ with this property can always be found for any partition of **N** into finitely many classes is completely open. For a complete and readable account of these and related developments, the reader should consult the survey of Hindman [Hi (79)a].

There is a rapidly growing body of results which has appeared recently and which goes under the name of Euclidean Ramsey Theory. The basic question it attacks is this: Given n and r, which configurations $C \subseteq \mathbf{E}^n$ have the property that for any partition of $\mathbf{E}^n$ into r classes, some class must contain a set *isometric* to C. For example, if C consists of 3 points forming a right triangle then a result of Shader [Shad (76)] shows that any partition of $\mathbf{E}^2$ into 2 classes always has a copy of C in at least one of the classes. A similar result also holds for 30° triangles and 150° triangles [Er + 5 (75)]. Note that this is not true if C is a unit equilateral triangle—in this case we simply partition the plane into alternating half open strips of width $\sqrt{3}/2$. The strongest conjecture dealing with this case is that for any partition of $\mathbf{E}^2$ into 2 classes, some class contains congruent copies of *all* 3-point sets with the possible exception of a single equilateral triangle.

A configuration $C \subseteq \mathbf{E}^n$ is called *Ramsey* if for all r, there is an $N(C, r)$ so that for any partition of $\mathbf{E}^N$ with $N \geqslant N(C, r)$ into r classes, some class always contains a subset congruent to C. There are two natural classes which are known to bound the Ramsey configurations. On one hand, it is known [Er + 5 (73)] that the set of the 2^n vertices of any rectangular parallelepided (= brick) is Ramsey (and consequently, so is every subset of a brick, e.g., every acute triangle). On the other hand, it is known [Er + 5 (73)] that every Ramsey configuration must lie on the surface of some sphere S^n. Thus, any set of 3 points in a straight line is not Ramsey (there are partitions of $\mathbf{E}^n$ into 16 classes which avoid having any particular 3 point linear set in one class). Thus, the Ramsey configurations lie between bricks and spherical sets. The unofficial consensus is that they are probably just the (subsets of) bricks but there is no strong evidence for this. Interesting special cases to attack here would be to decide if the vertices of an isosceles 120° triangle or the vertices of a regular pentagon are Ramsey.

Another result of this type more closely related to A.P.'s is the following. It has been shown that there is a large M so that it is possible to partition

$\mathbf{E}^2$ into two sets A and B so that A contains no pair of points with distance 1 and B contains no A.P. of length M. How small can M be made? The only estimate currently known is that $M \leqslant 10000000$ (more or less). In the other direction, it has just been shown by R. Juhász [Ju (79)] that we must have $M \geqslant 5$. In fact, she shows that B must contain a congruent copy of any 4-point set. As a final Euclidean Ramsey question, we mention the following. It was very recently shown by Graham [Gr (80)] (in response to a question of R. Gurevich [Bab (76)]) that for any r, there is a (very large) number $G(r)$ so that for any partition of the lattice points of the plane into r classes, some class contains the vertices of a right triangle with area *exactly* $G(r)$. It follows from this (see [Gr-Sp (79)] that for any partition of all the points of $\mathbf{E}^2$ into finitely many classes, some class contains the vertices of triangles of each area. The question is: Is this also true for rectangles? or perhaps parallelograms? On the other hand, it is certainly *not* true for rhombuses.

An interesting variation of van der Waerden's theorem is to require that the desired A.P. only hit one class more than the other class by some fixed amount (rather than be completely contained in one class). More precisely, let $f(n, k)$ denote the least integer so that if we divide the integers not exceeding $f(n, k)$ into two classes, there must be an A.P. of length n, say $a + ud$, $0 \leqslant u \leqslant n-1$, with $a + (n-1)d \leqslant f(n, k)$ such that

$$\sum_{u=0}^{n-1} g(a+ud) > k$$

where $g(m)$ is $+1$ if m is in the first class and -1 if m is in the second class. $f(2n, 0)$ has been determined by Spencer [Spen (73)] but we do not have a decent bound for even $f(n, 1)$. It seems likely that $\lim_n W(n)^{1/n} = \infty$ but perhaps $\lim_n f(n, cn)^{1/n} < \infty$. Unfortunately, we cannot even prove $\lim_n f(n, 1)^{1/n} < \infty$. Perhaps this will not be hard but we certainly do not see how to prove $\lim_n f(n, \sqrt{n})^{1/n} < \infty$. Define

$$F(x) = \min_g \max \left| \sum g(a+kd) \right|$$

where the maximum is taken over all A.P.'s whose terms are positive integers and the minimum is taken over all functions $g: \mathbf{Z} \rightarrow \{-1, 1\}$. Roth [Roth (64)] proved that

$$F(x) > cx^{1/4}$$

and conjectured that for every $\varepsilon > 0$, $F(x) > x^{1/2-\varepsilon}$ for $x > x_0(\varepsilon)$. In the other direction Spencer [Spen (72)] showed that

$$F(x) < cx^{1/2} \frac{\log \log x}{\log x} .$$

However, Sárközy (see [Er-Sp (74)]) subsequently showed that

$$F(x) = O\left((x \log x)^{1/3}\right),$$

disproving the conjecture of Roth.

Cantor, Erdös, Schreiber and Straus [Er (66)] (also see [Er (73) b]) proved that there is a $g(n) = \pm 1$ for which

$$\max_{\substack{a, m \\ 1 \leq b \leq d}} \left| \sum_{k=1}^{m} g(a+kb) \right| < h(d)$$

for a certain function $h(d)$. They showed that $h(d) < cd!$ No good lower bound for $h(d)$ is known. As far as we know the following related more general problem is still open. Let $A_k = \{a_1^{(k)} < a_2^{(k)} < \ldots\}$, $k = 1, 2, \ldots$ be an infinite class of infinite sets of integers. Does there exist a function $F(d)$ (depending on the sequences A_k) so that for a suitable $g(n) = \pm 1$

$$\max_{m, 1 \leq k \leq d} \left| \sum_{i=1}^{m} g(a_i^{(k)}) \right| < F(d) \ ?$$

It seems certain that the answer is affirmative.

Finally, is it true that for every c, there exist d and m so that

$$\left| \sum_{k=1}^{m} g(kd) \right| > c \ ?$$

The best we could hope for here is that

$$\max_{md \leq n} \left| \sum_{k=1}^{m} g(kd) \right| > c \log n .$$

We remark that these questions can also be asked for functions $g(n)$ which take kth roots of unity as values rather than just ± 1. However, very little is yet known for this case.

Another interesting problem: For $r < s$ denote by $f_r(n; s)$ the smallest integer so that every sequence of integers of n terms which contains $f_r(n; s)$ A.P.'s of length r must also contain an A.P. of length s. Perhaps for s

$= o(\log n)$, $f_3(n; s) = o(n^2)$; this is certainly false for $s > \varepsilon \log n$. At present we cannot even prove $f_3(n; 4) = o(n^2)$.

Abbott, Liu and Riddell [Ab-Li-Ri (74)] define $g_k(n)$ as the largest integer so that among any n real numbers one can always find $g_k(n)$ of them which do not contain an A.P. of length k. It is certainly possible to have $g_k(n) < r_k(n)$; in fact, Riddell shows that $g_3(14) = 7$ but $r_3(14) = 8$. It is not known if $g_3(n) < r_3(n)$ for infinitely many n. It follows from a very interesting general theorem of Komlós, Sulyok and Szemerédi [Kom-Su-Sz (75)] that $g_3(n) > cr_3(n)$. Perhaps $\lim_{n\to\infty} \frac{r_3(n)}{g_3(n)} = 1$. Szemerédi points out that it is not even known if $\frac{r_4(n)}{r_3(n)} \to \infty$.

The following question is due to F. Cohen. Determine or estimate a function $h(d)$ so that if we split the integers into two classes, at least one class contains for infinitely many d an A.P. of difference d and length at least $h(d)$. Erdös observed that $h(d) < cd$ is forced and Petruska and Szemerédi [Pe-Sz (∞)] strengthened this by showing that $h(d) < cd^{1/2}$. Very recently, J. Beck [Bec (xx)] showed $h(d) < \frac{(1+o(1)) \log d}{\log 2}$. The theorem of van der Waerden shows that $h(d) \to \infty$ with d but we currently have no usable lower bound for $h(d)$.

Define $H(n)$ to be the smallest integer so that for any partition of the integers $\{1, 2, \ldots, H(n)\}$ into any number of disjoint classes, there is always an n-term arithmetic progression all of whose terms either belong to one class or all different classes. The existence of $H(n)$ is guaranteed by Szemerédi's theorem. In fact it is easy to show $H(n)^{1/n} \to \infty$; to show $H(n)^{1/n}/n \to \infty$ might be much harder. What can be said about small values of $H(n)$?

Is it true that for any partition of the pairs of positive integers into two classes, the sums $\sum_{x \in X} \frac{1}{\log x}$ are unbounded where X ranges over all subsets which have all pairs belonging to one class?

It was conjectured by Erdös that for every $\varepsilon > 0$ there is a t_ε so that the number of squares in any A.P. $a + kd$, $0 \leqslant k \leqslant t_\varepsilon$, is less than $\varepsilon t_\varepsilon$. This follows from Szemerédi's result $r_k(n) = o(n)$; in fact, his earlier result $r_4(n) = o(n)$ (see [Sz (69)]) suffices for this purpose. Rudin conjectured that there is an absolute constant c so that the number of squares in $a + kd$, $0 \leqslant k \leqslant t$, is less than $c\sqrt{t}$. Rudin's conjecture is still open.

Denote by $F(n)$ the largest integer r for which there is a non-averaging sequence $1 \leqslant a_1 < \ldots < a_r \leqslant n$, i.e., no a_i is the arithmetic mean of other a_j's. Erdös and Straus [Er-Str (70)] proved

$$\exp(c\sqrt{\log n}) < F(n) < n^{2/3}.$$

However, Abbott [Ab (75)] just proved the unexpected result

$$F(n) > n^{1/10}.$$

It would be nice to know what the correct exponent is here.

It seems to be difficult to state reasonable conditions which imply the existence of an infinite A.P. in a set of integers. For example, because there are only countably many infinite A.P.'s, then for any sequence a_n, there is a sequence b_n with $b_n > a_n$ so that the b_n's hit every infinite A.P. It is not difficult to show that for any sequence $B = (b_1, b_2, \ldots)$ with $b_1 \geqslant 5$ and $b_{i+1} \geqslant 2b_i$ there is a set $A = \{a_1, a_2, \ldots\}$ with $2 \leqslant a_{k+1} - a_k \leqslant 3$ for all k so that for all i, $b_i \notin A + A = \{a + a' : a, a' \in A\}$. Whether such behavior can hold for $A + A + A$ (or more summands) is not known.

The situation is completely different, however, when one considers *generalized* A.P.'s

$$S(\alpha, \beta) = (a_1, a_2, \ldots, a_n, \ldots).$$

A generalized A.P. is formed by $a_n = [\alpha n + \beta]$ for given real $\alpha \neq 0$ and β. It follows from results of Graham and Sós [Gr-Só (xx)] that if $b_{n+1}/b_n \geqslant c > 2$ then the complement of the b_k's contains an infinite generalized A.P. This has very recently been strengthened by Pollington [Poll (xx)] who proved that there is no sequence b_n hitting every generalized A.P. with $b_{k+1}/b_k \geqslant c > 1$ for all k. On the other hand, for any sequence c_n there exist sequences b_n with $b_n > c_n$ which hit every generalized A.P.

Of course, almost any question dealing with A.P.'s can also be asked about generalized A.P.'s. For example, can we get better (much better?) bounds for van der Waerden's theorem when we allow generalized A.P.'s? This question has not yet been investigated so far.

The generalized A.P.'s $S(\alpha) = S(\alpha, 0) = \{[\alpha n] : n = 1, 2, \ldots\}$ have an extensive literature (e.g., see [Frae (69)], [Ni (63)], [Frae-Le-Sh (72)], [Gr-Li-Li (78)] and especially [Stol (76)]). One of the earliest results [Bea (26)] asserts that $S(\alpha_1)$ and $S(\alpha_2)$ disjointly cover the positive integers iff the α_i are irrational and $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1$. An old result of Uspensky [U (27)], [Gr (63)] shows that $\mathbf{Z}^+$ can never be partitioned into three or more disjoint $S(\alpha_i)$; in fact, for any three $S(\alpha_i)$, some pair of them must have infinitely many common elements.

This situation does not hold for general $S(\alpha, \beta)$ however. For example, $S(2;0), S(4;1), S(4;3)$ and $S\left(\frac{7}{4};0\right)$, $S\left(\frac{7}{2};\frac{5}{2}\right)$, $S(7;4)$ both form decompositions of the nonnegative integers. Of course, more generally, if

$$\mathbf{Z} = \sum_{i=1}^{m} S(a_i; b_i), \quad a_i, \ b_i \in \mathbf{Z}$$

is a decomposition of $\mathbf{Z}$ into disjoint A.P.'s of integers then $S(\alpha; \beta) = \sum_{i=1}^{m} S(a_i\alpha; \beta + \alpha b_i)$. It has been shown by Graham [Gr (73)] that if $m \geqslant 3$, $\mathbf{Z}^+ = \sum_{i=1}^{m} S(\alpha_i; \beta_i)$ and some α_i is irrational then the $S(\alpha_i; \beta_i)$ must be generated from two disjoint $S(\gamma_i; 0)$ which cover $\mathbf{Z}^+$ by transformations of this type. In particular, it follows from the theorem of Mirsky and Newman (see [Er (50)]) that for some $i \neq j$, $\alpha_i = \alpha_j$. Curiously, the situation is much less well understood when all the α_i are rational. A striking conjecture of Fraenkel [Frae (73)] asserts that for any such decomposition (with $m \geqslant 3$) with all α_i distinct, we must have $\{\alpha_1, ..., \alpha_m\} = \left\{\frac{2^m - 1}{2^k} : 0 \leqslant k < m\right\}$.

One can ask how sparse (in some sense) a set S of integers can be and still have the property that for any decomposition of S into r classes, some class must contain an A.P. of length k. Of course, since the multiples of any d have this property, we must be more precise about what we mean by sparse. For example, we might ask whether such an S exists which itself contains no A.P. of length $k + 1$. That such S's exist was first shown by Spencer [Spen (75)] (using the previously mentioned theorem of Hales and Jewett) and independently by Nešetřil and Rödl [Neš-Röd (∞)].

An old theorem of Brauer [Bra (28)] (also see [Ab-Ha (72)], [Rad (33) a]) proves a stronger form of van der Waerden's theorem in which not only must one of the classes contain an A.P. of length r, say, $a + kd$, $0 \leqslant k < r$, but also the common difference d as well. However, the analogue of Szemerédi's theorem does not hold for this case—we can find sets of positive density which do not contain a k-term A.P. together with its difference. For example, the set of odd integers cannot contain a, $a + d$ and d. However, the densest subset R of $\{1, 2, ..., n\}$ not containing a k-term A.P. and its difference has recently been determined by Graham, Spencer and Witsenhausen [Gr-Sp-Wi (77)]. Their result shows that any such R must satisfy $|R| \leqslant n - \left[\frac{n}{k}\right]$ (and this is best possible). Almost all cases of this type of problem remain open. One of the simplest is this: Let R_n be a maximum subset of $\{1, 2, ..., n\}$ with the property that for no x are x, $2x$ and $3x$ all in R_n. What is $\lambda = \lim_n \frac{|R_n|}{n}$? (Its existence is known). In particular, prove that λ is irrational. Of course, one could ask these questions for infinite sets of integers. For example, if $a_1 < a_2 < ...$ is an infinite sequence of integers such that for no x are x, $2x$, $3x$ all a_i's, then how large can the density of the a's be (if it exists)? Can the upper density be larger?

In a different direction, one could ask how many subsets of $\{1, 2, ..., n\}$, say $S_1, ..., S_t$, can one have so that for all $i \neq j$, $S_i \cap S_j$ is an A.P. Simonovits, Sós and Graham [Gr-Si-Só (80)] have recently shown that $t \leqslant \binom{n}{3} + \binom{n}{2} + \binom{n}{1} + 1$ and this is best possible. If $S_i \cap S_j$ must be a *nonempty* A.P. then Simonovits and Sós [Sim-Sós (xx)] have given an ingenious proof that $t < cn^2$. It is conjectured in this case that the maximum families form strong Δ-systems, i.e., the S_i are just all the finite A.P.'s in $\{1, 2, ..., n\}$ containing a particular element, presumably the integer $\left[\frac{n}{2}\right]$.

An easy consequence of van der Waerden's theorem is the following: If $A = (a_1, a_2, ...)$ is an increasing infinite sequence of integers with $a_{k+1} - a_k$ bounded then A contains arbitrarily long A.P.'s (see [Kak-Mo (30)]). The analogous questions in higher dimensions are not yet completely settled. For example, let $p_i = (x_i, y_i)$, $i = 1, 2, ...$ be an infinite set of lattice points in $\mathbf{E}^2$ so that $p_{i+1} - p_i$ is either (0, 1) or (1, 0). Must the p_i contain arbitrarily long A.P.'s? Surprisingly, the answer is no. It is possible to use the strongly non-repetitive sequences of Dekking [Dek (79)] (also, see Pleasants [Ple (70)], [Bro (71)]) to construct such a sequence of p_i having no 5-term A.P. On the other hand, it is not hard to see that 4-term A.P.'s cannot be avoided. Similar techniques can be used to show that there are increasing unit-step sequences of lattice points in $\mathbf{E}^5$ containing no 3-term A.P. Whether or not this can be done in $\mathbf{E}^3$ or $\mathbf{E}^4$ is not known. Pomerance [Pom (xx)] has recently shown that if the *average* step size is bounded, there must be arbitrarily large sets of the a_k which lie on some line. In fact, he shows somewhat more, e.g., that the same conclusion holds for the points

(n, p_n) where p_n denotes the n^{th} prime [Pom (79)] (however, the proof of the former does not provide effective bounds).

Gerver and Ramsey [Ge-Ra (xx)] give an effective estimate for the following special case. Suppose $S \subseteq \mathbf{Z}^2$ is finite and let $A = (a_1, a_2, ..., a_N)$ be a sequence of lattice points with $a_{k+1} - a_k \in S$ for all $k < N$. (Such a sequence is called an *S-walk*). Then for any $\varepsilon > 0$, if $N \geqslant N_0(M, \varepsilon)$ where M denotes the maximum distance of any point in S from the origin, A must contain at least $C(M, \varepsilon)(\log N)^{1/4-\varepsilon}$ collinear points. On the other hand, such a result does not hold for $\mathbf{Z}^3$. In particular, they construct an infinite sequence $B = (b_1, b_2, ...)$ of lattice points in $\mathbf{Z}^3$ for which $b_{k+1} - b_k$ is a unit vector for all k and such that B has at most 5^{11} collinear points. They conjecture that 3 is actually the correct bound for their construction. It is not known whether there is an infinite S-walk in $\mathbf{Z}^3$ for S finite which has no three points collinear.

A number of interesting questions involving A.P.'s come up in the following way. Let us say that a (possibly infinite) sequence $(a_1, a_2, ...)$ has a *monotone* A.P. of length k if for some choice of indices $i_1 < i_2 < ... < i_k$, the subsequence $a_{i_1}, a_{i_2}, ..., a_{i_k}$ is either an increasing or a decreasing A.P. It has often been noted that it is possible to arrange any finite set of integers into a sequence containing no monotone A.P. of length 3. Essentially, this can be done by placing all the odd elements to the left of all the even elements, arranging (by induction) the odds and the evens individually to have no monotone A.P.'s of length 3 and using the fact that the first and last terms of a 3-term A.P. must have the same parity. If $M(n)$ denotes the number of permutations of $\{1, 2, ..., n\}$ having no monotone 3-term A.P., it has been shown by Davis, Entringer, Graham and Simmons [Dav+3 (77)] that

$$M(n) \geqslant 2^{n-1}\,, \quad M(2n-1) \leqslant (n!)^2\,, \quad M(2n+1) \leqslant (n+1)\,(n!)^2$$

It would be interesting to know if $M(n)^{1/n}$ is bounded or even tends to a limit. The situation for permutations of infinite sets is different. It has been shown by the above mentioned authors that any permutation of $\mathbf{Z}^+$ contains an increasing 3-term A.P. but that there exist permutations of $\mathbf{Z}^+$ which have no monotone 5-term A.P.'s. The question of whether or not monotone 4-term A.P.'s must occur is currently completely open. If one is allowed to arrange $\mathbf{Z}^+$ into a doubly-infinite sequence $..., a_{-1}, a_0, a_1, ...$ then monotone 3-term A.P.'s must still occur but it is now possible to prevent those of length 4. If the elements to be permuted are *all* the integers rather than just the positive integers, less is known. It is known [Odd (75)] that monotone 7-term A.P.'s can be stopped in the singly-infinite case. We should note that the modular analogues of these problems have been studied by Nathanson [Na (77)a].

Must any ordering of the reals contain a monotone k-term arithmetic progression for every k?

We conclude this topic with a very annoying question: Is it possible to partition $\mathbf{Z}^+$ into two sets, each of which can be permuted to avoid monotone 3-term A.P.'s? If we are allowed three sets, this is possible; the corresponding situation for $\mathbf{Z}$ has not been investigated.

It is not difficult to find finite sets $A = \{ a_1, ..., a_n \}$ with the property that for any two elements $a_i, a_j \in A$, there is an $a_k \in A$ so that $\{ a_i, a_j, a_k \}$ forms an A.P., e.g., $\{ 1, 2, 3 \}$ and $\{ 1, 3, 4, 5, 7 \}$. In fact, it is not difficult to show that up to some affine transformation $x \to ax + b$, these are the only such sets. It follows from this that the analogue of Sylvester's theorem holds for A.P.'s, i.e., no finite set A has the property that every 3 terms of A belong to some A.P. in A. Suppose one only requires that for every choice of k terms from A, some 3 (or m) of them belongs to an A.P. in A. Can those A now be characterized? One might also ask these questions for generalized A.P.'s as well where we would expect much richer classes of A's because of the greater number of generalized A.P.'s.

Stanley has raised the following question (generalizing an earlier question of Szekeres (see [Er-Tu (36)])). Starting with $a_0 = 0$, $a_1 = a$, form the infinite sequence $a_0, a_1, a_2, a_3, ...$ recursively by choosing a_{n+1} to be the least integer exceeding a_n which can be adjoined so that no 3-term A.P. is formed. Can the a_k be explicitly determined? For example, if $a = 1$ then the a_k are just those integers which have no 2 in their base 3 expansion. Similar characterizations are known when $a = 3^r$ and $a = 2 \cdot 3^r$ (see [Odl-Sta (78)]). For these cases, if $\alpha = \frac{\log 3}{\log 2}$ then $\liminf_n \frac{a_n}{n^\alpha} = 1/2$, $\limsup_n \frac{a_n}{n^\alpha} = 1$. However, the case of $a = 4$ (and all other values not equal to 3^r or $2 \cdot 3^r$) seems to be of a completely different character. There are currently no conjectures for the a_k in this case.

Hoffman, Klarner and Rado [Kl-Ra (73)], [Kl-Ra (74)], [Hoff-Kl (78)], [Hoff-Kl (79)], [Hoff (76)] have obtained many interesting results on the following problem: Let R denote a set of linear operations on the set of nonnegative integers, each of the type $\rho(x_1, ..., x_r) = m_0 + m_1x_1 + ... + m_rx_r$. Given a set A of positive integers, let $< R : A >$ denote the smallest set containing A which is closed under all operations in R. What is the structure of $< R : A >$? Two basic results here are:

(i) For any infinite set B there is a finite set A such that $< R : B > = < R : A >$ whenever at least one $\rho(x_1, \ldots, x_2) = m_0 + m_1 x_1 + \ldots + m_r x_r$ has $(m_1, \ldots, m_r) = 1$.

(ii) If $R = \{ m_0 + m_1 x_1 + \ldots + m_r x_r \}$ and all m_i are positive then $< R : A >$ is a finite union of infinite A.P.'s, again when $(m_1, \ldots, m_r) = 1$.

The special case that $R = \{ a_1 x + b_1, \ldots, a_r x + b_r \}$ is particularly interesting. It has been shown by Erdös, Klarner and Rado (see [Kl-Ra (74)]) that if $\sum_{k=1}^{r} \frac{1}{a_k} < 1$ then $< R : A >$ has density 0. The situation in which $\sum_{k=1}^{r} \frac{1}{a_k} = 1$ is not yet completely understood. This depends, for example, on when the set of transformations $x \to a_i x + b_i$, $1 \leqslant i \leqslant r$, generates a free semigroup under composition. The reader should consult the relevant references for numerous other results and questions.

Erdös asked: If S is a set of real numbers which does not contain a 3-term A.P. then must the complement of S contain an infinite A.P.? R. O. Davies (unpublished) showed that assuming the Continuum Hypothesis the answer is no; Baumgartner [Bau (75)] proved the same thing without assuming the Continuum Hypothesis. Baumgartner also proved the conjecture of Erdös that if A is a sequence of positive integers with all sums $a + a'$ distinct for $a, a' \in A$ then the complement of A contains an infinite A.P. Of course, many generalizations are possible.

Can one prove that the longest arithmetic progression

$$\{ a + kd : 0 \leqslant k \leqslant t \}$$

with $a + kt < x$, which consists entirely of primes satisfies $t = o(\log x)$? Only $t \leqslant (1 + o(1)) \log x$ is clear; this follows from the prime number theorem. Suppose that at least ct of the terms are prime. It is not hard to see that $t < (\log x)^{\alpha(c)}$ where $\alpha(c) \to \infty$ as $c \to o$. If there is any justice $\alpha(c)$ should not tend to infinity. If we take $t = \log x$ perhaps the number of primes tends to 0 uniformly in d.

3. Covering congruences

A family of residue classes $a_i \pmod{n_i}$ with $1 < n_1 < \ldots < n_r$ is called a *system of covering congruences* if every integer belongs to at least one of the residue classes, i.e., every integer satisfies at least one of the congruences $x \equiv a_i \pmod{n_i}$. In 1934 Romanoff asked P. Erdös if there are infinitely many odd integers not of the form $p + 2^k$ where p is prime. This led Erdös [Er (50) a] to the concept of covering congruences and he answered Romanoff's question affirmatively by using the system [1]) of covering congruences

$$0 \pmod 2,\ 0 \pmod 3,\ 1 \pmod 4,\ 3 \pmod 8,\ 7 \pmod{12},\ 23 \pmod{24}\,.$$

The major open problem in this topic is:

> Is it true that for every c there is a system of covering congruences with $n_1 \geqslant c$?

The current record is held by Choi [Cho (71)] who constructed covering congruences with $n_1 = 20$. If the answer is affirmative we immediately obtain that for every m there is an arithmetic progression no term of which is a sum of a power of two and an integer having at most m prime factors.

Is it true that there is a system of covering congruences with all n_i odd, or more generally, relatively prime to a given integer d? Selfridge (see [Sch (67) b]) proved the interesting result that a system of covering congruences with all odd moduli exists if a covering system exists with no n_i dividing any other n_j. Schinzel [Sch (67) b] applies these ideas to certain polynomial irreducibility problems. Can one choose all the n_i to be of the form $p - 1$ for p prime and at least 5? If $p = 3$ is allowed then Selfridge has given such an example using the divisors of 360.

Denote by $f(u)$ the smallest integer so that there is a system of $f(u)$ covering congruences with $n_1 = u$. If $f(u)$ is finite can we obtain reasonable estimates of it? It should be quite large, e.g., it is probably true that

$$\frac{f(u)}{u^k} \to \infty \text{ for every } k\,.$$

(where a system of covering congruences is always assumed to be finite unless stated otherwise). It should be true that

[1]) No residue class mod 6 could be used since 6 is the only integer n such that $2^n - 1$ has no primitive prime divisor.

$$\min_{u=n_1<n_2<\dots} \sum_i \frac{1}{n_i} \to \infty$$

as $u \to \infty$. If so, how fast does the sum tend to infinity?

Denote by $A(n)$ the largest number of disjoint systems of covering congruences which can be formed using all moduli less than or equal to n. Estimate $A(n)$ from above and below. Of course we do not yet even know that $A(n) \to \infty$.

Let $B(n)$ denote the largest integer so that for a suitable choice of a_i, every integer satisfies at least $B(n)$ of the congruences $a_i \pmod i$ for $1 \leqslant i \leqslant n$. What is the relation between $A(n)$ and $B(n)$? For integers $n < m$ let $A(m, n)$ denote the least possible density of integers not covered by the congruences $a_i \pmod{(n+i)}$, $1 \leqslant i \leqslant m - n$, taken over all choices of a_i. Trivially

$$1 - \sum_i \frac{1}{n+i} < A(m, n) < \prod_i \left(1 - \frac{1}{n+i}\right).$$

This should be improved if m/n is large. Is it true that $A(m, n) > \varepsilon_c$ if $m/n < c$ (for n large)?

Let $0 < n_1 < n_2 < \dots < n_r$. It would be nice to get nontrivial conditions on the n_i which would guarantee that a system of covering congruences $\{a_i \pmod{n_i}\}$ exists. Is it true that if $\sum_{n_i<x} \frac{1}{n_i}$ grows rapidly enough then such a system exists? The same question applies if we consider the growth of $\sum_{n_i<x} 1$. It seems certain that $\sum_{n_i<x} 1 = x + o(x)$ will be needed, i.e., there is no system of covering congruences where all the moduli are between εx and x. Determine or estimate the largest $h(x)$ so that there is a system of covering congruences with $n_1 = h(x)$ and $n_r \leqslant x$. Our ignorance is complete here—on one hand, $h(x)$ could be bounded; on the other hand even $h(x) > \varepsilon x$ cannot be excluded.

Is it true that if the positive integers are partitioned into finitely many classes then at least one of the classes contains the moduli of a covering system? Perhaps if a subset $X \subseteq \mathbf{Z}^+$ has positive upper density then X already must contain the moduli of a covering system.

Some time ago Erdös conjectured and L. Mirsky and M. Newman proved (e.g., see [Z (69)], [Er (50) a] or [Er (52)]) that there is no system of *exact* covering congruences, i.e., a system $\{a_i \pmod{n_i}\}$ with $n_1 < n_2 < \dots < n_r$ such that every x satisfies *exactly one* of the congruences $x \equiv a_i \pmod{n_i}$. In fact, it is not hard to show that if $\{a_i \pmod{n_i}\}$ covers the integers exactly with $n_1 \leqslant n_2 \leqslant ... \leqslant n_r$ then $n_{r-1} = n_r$. Exact covering systems have a fairly large literature so we will just restrict ourselves to mentioning some of the main references (see [Frae (73)], [Z (69)], [New (71)], [Kruk (71)], [Dew (72)], [Z (74)], [Er (71) *]).

The following recent problem of Herzog and Schönheim should be mentioned here: If G is an abelian group, can there exist an exact covering of G by cosets of different sizes?

A system of congruences is called *disjoint* if no integer satisfies more than one of them. Erdös and Stein conjectured that if $\{ a_i \pmod{n_i} \}$ with $n_1 < ... < n_r \leqslant x$ is a disjoint system of congruences then $r = o(x)$. This was proved by Erdös and Szemerédi [Er-Sz (68)] who, in fact, showed that if $f(x)$ denotes the maximum possible value of r above then

$$\frac{x}{\exp\left((\log x)^{1/2+\varepsilon}\right)} < f(x) < \frac{x}{(\log x)^c}.$$

It seems likely that the lower bound is closer to the truth but it does not seem to be easy to prove this.

Erdös conjectured that if a system $\{ a_i \pmod{n_i},\ 1 \leqslant i \leqslant r \}$ covers 2^r consecutive integers then it covers all integers. This was proved by Selfridge and also by Crittenden and Vanden Eynden [Cr-VE (69)], [Cr-VE (70)]. This bound is best possible as the system $\{ 2^{i-1} \pmod{2^i},\ 1 \leqslant i \leqslant r \}$ shows.

Suppose $\{ a_i \pmod{n_i} \}$ is a covering system and assume each n_i has a prime factor exceeding k. What estimates can be made for $\sum_i \frac{1}{n_i}$?

In [Gr (64) e], Graham considers the following question: Is there an infinite "Lucas" sequence $a_0, a_1, ...$ satisfying $a_{n+2} = a_{n+1} + a_n$, $n \geqslant 0$, and $(a_0, a_1) = 1$ such that no a_n is prime? The starting choice $a_0 = 0$, $a_1 = 1$ generates the familiar Fibonacci numbers; it is conjectured that infinitely many of these are prime but a proof of this at present seems hopeless. The recent doctoral dissertation of C. Stewart [Stew_2 (75)], [Stew_2 (76)] contains the strongest results currently available in this direction.

It turns out that such composite Lucas sequences exist. By using a system of covering congruences, it was shown [Gr (64) e] that the following choice generates such a sequence:

$$a_0 = 1786772701928802632268715130455793,$$
$$a_1 = 1059683225053915111058165141686995.$$

This is the smallest pair (a_0, a_1) which is known to work. It would be very surprising if a pair existed with both components less than 10^{20}. Is it possible for a Lucas sequence to have all terms composite *without* having an underlying system of covering congruences responsible? (In other words, no positive integer has a common factor with every term of the sequence).

It is well known (see [Bat (63)]) that there are odd integers $2m + 1$ so that none of the numbers $2^k(2m+1) + 1$ is a prime. The smallest such number is not known; it is $\geqslant 3061$ and $\leqslant 78557$. ([Rob (58)], [Me (76)], [Self (76)]). H. C. Williams [Wil (xx)] recently eliminated the long-time contender 383 by showing that $383 \cdot 2^{6393} + 1$ is prime. Sierpiński [Sie (60)] showed that these numbers $2m + 1$ have positive lower density. On the other hand, Erdös and Odlyzko [Er-Odl (xx)] recently proved that the lower density of the complementary set is positive.

Are there integers m with $(m, 6) = 1$ so that none of the numbers $2^\alpha 3^\beta m + 1$ is prime? What about for $p_1^{\alpha_1} \dots p_r^{\alpha_r} m + 1$? What about for $q_1 \dots q_r m + 1$ where the q_i are primes congruent to 1 (mod 4)? If $2^\alpha t + 1$ is never prime for a fixed odd t where $\alpha = 1, 2, \dots$, must there be a covering system responsible, i.e., must there be an $N > 0$ so that $(2^\alpha t + 1, N) > 1$ for all $\alpha > 0$? The answer is probably no since otherwise, for example, this would imply that there are infinitely many Fermat primes, i.e., primes of the form $2^{2^t} + 1$. This type of problem can be posed in many forms but it always seems hopeless.

In [Ben-Er (74)] it was asked if there is a constant C so that if $\dfrac{\sigma(n)}{n} > C$ then the divisors of n can be used as the moduli of a system of covering congruences. Very recently J. Haight [Hai (79)] has shown that no such C exists.

For which n is it possible to form a covering system $a_d \pmod{d}$ where $d \mid n$ which is as disjoint as possible, i.e., so that if

$$x \equiv a_d \pmod{d}, \quad x \equiv a_{d'} \pmod{d'}$$

then $(d, d') = 1$? The density of such n is zero. For a given n what is the minimum density of the integers which do not satisfy any of the congruences? Probably no such n exists if the system is required to be a covering system.

For every integer n there is a real number c_n defined as follows: For all divisors $d_i > 1$ of n, form all possible congruences $a_i \pmod{d_i}$, $1 < d_1 < \dots < d_t = n$. Let c_n be the greatest lower bound of the densities of the set of integers not satisfying any of these congruences. The density of integers n with $c_n = 0$ exists and the c_n have a distribution function which is continuous (except at 0) and is strictly monotonic. If $c_n = 0$, n is called *covering*. Every such number is a multiple of a "primitive" covering number n'. No doubt, the sum $\sum \frac{1}{n'}$ taken over all primitive covering numbers converges.

For a finite set of moduli $n_1, n_2, \ldots, n_r$, one can ask for the minimum value of the density of integers not hit by a suitable choice of congruences $a_i \pmod{n_i}$. Is the worst choice obtained by taking all the a_i equal?

We next return to several questions related to the original question of Romanoff which motivated the concept of covering congruences. Denote by $V(n)$ the number of prime factors of n with multiple factors counted multiply. Is it true that all large integers are of the form $2^k + m$ where $V(m) < \log \log m$? It is easy to see by probabilistic methods that this holds for almost all numbers. Perhaps $\log \log m$ can be replaced by $\varepsilon \log \log m$ or even some function which tends to infinity much more slowly. Cohen and Selfridge [Coh-Se (75)] found an infinite arithmetic progression of odd numbers none of which is the sum or differences of two prime powers (and consequently not the sum or difference of a prime and a power of 2).

Is it true that for every $r \geqslant 2$ there are infinitely many integers not the sum of a prime and at most r powers of 2? Is there an infinite arithmetic progression of such numbers? For $r = 2$, Crocker [Cro (71)] proved that there are infinitely many such numbers but he does not get an arithmetic progression. Gallagher [Gal (75)] has shown that for every $\varepsilon > 0$ there is an r so that the lower density of the integers of the form $p + 2^{k_1} + \ldots + 2^{k_r}$ exceeds $1 - \varepsilon$. Is it true that all (or almost all) integers are the sum of a power of 2 and a squarefree number?

It is possible to extend the concept of covering systems to include the possibility of infinite systems of congruences. However, the situation is not completely satisfactory here since there are several competing definitions for infinite systems of covering congruences. We will discuss several of these now although it is certainly possible that we overlook trivial observations.

To begin with, we could call an infinite system $\{ a_i \pmod{n_i} \}$ *covering* if every integer satisfies at least one of them and the density of integers not satisfying the first k tends to 0 as $k \to \infty$. If $\sum_i \frac{1}{n_i} = \infty$ this can always be done so that the only interesting case is when $\sum_i \frac{1}{n_i} < \infty$. As in the case of ordinary (finite) covering systems, we can ask whether a set of positive density always contains the moduli of an infinite covering system.

Alternatively, one could define an infinite system $\{ a_i \,(\text{mod}\; n_i) \}$ to be covering if every (large) integer is of the form $a_i + tn_i$, $t \geqslant 1$. This prevents every sequence of n_i's from being the moduli of an infinite covering system. If $\sum_i \frac{1}{n_i} = \infty$ then almost all integers can be of the form $a_i + tn_i$, $t \geqslant 2$, but this is certainly not the case for *all* large integers. More generally, one could define $\{ a_i \,(\text{mod}\; n_i) \}$ to be a covering system if for some k, all but a finite number of positive integers are of the form $a_i + tn_i$ for some $t \geqslant k$. Is it true that with this definition, the primes form the moduli of an infinite covering system for every k? Even the case $k = 3$ already seems to be difficult. It seems that if

$$\sum_{n_i < x} \frac{1}{n_i} - \log \log x \to \infty$$

and

$$\sum_{n_i < x} 1 > cx/\log x$$

then there are choices of a_i so that $\{ a_i \,(\text{mod}\; n_i) \}$ is a covering system of this type.

Still another possibility (suggested by Selfridge) is this. The infinite system $\{ a_i \,(\text{mod}\; n_i) \}$ is said to be covering if when $f(k)$ denotes the number of integers $m < n_k$ with $m \not\equiv a_i \,(\text{mod}\; n_i)$, $1 \leqslant i \leqslant k$, then $f(k)/k \to 0$ as $k \to \infty$. As before, it is not clear if every sequence of positive density contains the moduli of a covering system of this type. Probably if $n_k > (1+\varepsilon)\, k \log k$ for $\varepsilon > 0$ and every k then $\{ a_i \,(\text{mod}\; n_i) \}$ is never a covering system of this type but this is not known and may have to await improvements in current sieve methods.

Suppose $n_1 < n_2 < \ldots$ are such that for every choice of a_i the set of integers not satisfying any of the congruences $a_i \,(\text{mod}\; n_i)$ has density 0. In this case we must have $\sum_i \frac{1}{n_i} = \infty$ and, if the n_i are pairwise relatively prime, then this suffices. This property clearly holds if for every ε there is a k so that the density of integers not satisfying $a_i \,(\text{mod}\; n_i)$, $1 \leqslant i \leqslant k$, is less than ε. Is this in fact necessary?

4. Unit fractions

One of the most ancient problems in mathematics is the representation of rationals $\frac{a}{b}$ in the form $\sum_{i=1}^{n} \frac{1}{x_i}$ with $x_1 \leqslant x_2 \leqslant ... \leqslant x_n$. For reasons which are not entirely clear (to us) the Egyptians considered fractions of the form $\frac{1}{m}$ to be much simpler than the general expression $\frac{a}{b}$. Perhaps the first result in the subject was due to Leonardo Pisano (= Fibonacci) [Pis (1857)] in 1202. He proved that the "greedy" algorithm can always be used to express any positive rational $\frac{a}{b}$ as a finite sum of distinct unit fractions, where with the greedy algorithm, we always choose the *largest* unit fraction $\frac{1}{m}$ not yet used for which the remainder is nonnegative.

Both of the authors have written a number of papers on unit fractions. Without claiming completeness, we will state many problems and results on this topic.

To begin with, we state an old question of Stein [Stei (58)]: In representing $\frac{a}{2b+1}$ as a sum of distinct unit fractions of the form $\frac{1}{2m+1}$, does the greedy algorithm always terminate? It is known that it is always possible to represent $\frac{a}{2b+1}$ as a sum of distinct odd unit fractions (e.g., see [Gr (64) a], [Al-Li (63)], [Stew$_1$ (54)], [Bre (54)]). More generally, it has been shown by the second author [Gr (64) a] that $\frac{a}{b}$ can be expressed as a sum of distinct unit fractions of the form $\frac{1}{pm+q}$ if and only if $\left(\frac{b}{(b,(p,q))}, \frac{p}{(p,q)}\right) = 1$. One could also ask whether the greedy algorithm always terminates in any of these cases as well.

The situation can change if we perturb the set of allowable denominators slightly. For example, define $u_1 = 1$ and $u_{n+1} = u_n(u_n+1)$, $n \geqslant 1$, and let $S = \{ n > 0 : n \neq u_k, k \geqslant 1 \}$. Then it can be shown that the set of rationals for which the greedy algorithm does not terminate when using only denominators in S is dense in $\mathbf{R}^+$ (although every positive rational has infinitely many representations as a sum of distinct reciprocals from S).

In a similar vein, it is known [Gr (64) d] that $\frac{a}{b}$ can be written as a finite sum of reciprocals of distinct squares if and only if $\frac{a}{b} \in \left[0, \frac{\pi^2}{6} - 1\right) \cup \left[1, \frac{\pi^2}{6}\right) = I$. It seems likely that there are rationals in this range for which the corresponding greedy algorithm does not terminate. Perhaps this is so for almost all the rationals in I.

A classical (and often rediscovered) result of Curtiss [Cu (22)] states that the closest strict underapproximation R_n of 1 by a sum of n unit fractions is always given by taking $R_n = \sum_{k=1}^{n} \frac{1}{u_k + 1}$ where u_k was defined previously. In this case

$$1 - R_n = \frac{1}{u_{n+1}}$$

so that R_n is also formed by a greedy algorithm, i.e., by choosing for the next term the largest unit fraction whose subtraction leaves a positive remainder. The analogous fact is also known [Er (50) b] to hold for under-approximations of rationals of the form $\frac{1}{m}$. Although the corresponding result does not hold for arbitrary rationals (e.g., $R_1\left(\frac{11}{24}\right) = \frac{1}{3}$, $R_2\left(\frac{11}{24}\right) = \frac{1}{4} + \frac{1}{5}\Big)$, it does always hold eventually. In other words, it is true that for any rational $\frac{a}{b}$, the closest strict underapproximation $R_n\left(\frac{a}{b}\right)$ of $\frac{a}{b}$ by a sum of n unit fractions is given by

$$R_n\left(\frac{a}{b}\right) = R_{n-1}\left(\frac{a}{b}\right) + \frac{1}{m}$$

where m is the least denominator not yet used for which $R_n\left(\frac{a}{b}\right) < \frac{a}{b}$, provided that n is sufficiently large. An attractive conjecture is that this also holds for any algebraic number as well. It is not difficult to construct irrationals for which the result fails. Conceivably, however, it holds for almost all reals.

For each n, let $\mathscr{X}_n$ denote the set

$$\left\{ \{ x_1, \ldots, x_n \} : \sum_{k=1}^{n} 1/x_k = 1 \, , \; 0 < x_1 < \ldots < x_n \right\}$$

and let $\mathscr{X}$ denote $\bigcup_{n \geq 1} \mathscr{X}_n$. Similarly, let $\mathscr{X}'_n$ and $\mathscr{X}'$ denote the corresponding sets when we only require $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$. There are many attractive unresolved questions concerning these sets, a few of which we now mention. Usually we will just state the problem for $\mathscr{X}_n$ and omit the corresponding statement for $\mathscr{X}'_n$.

To start with it would be interesting to have asymptotic formulas or even good inequalities for $| \mathscr{X}_n |$. The only estimates currently known are due to Straus and the authors. These are

$$e^{n^{2-\varepsilon}} < | \mathscr{X}_n | \leqslant c_0^{2^{n+1}}$$

where $c_0 = \lim_n u_n^{1/2^n} = 1.264085\ldots$ (see [Ah-Sl (73)]). Perhaps the lower bound can be replaced by $c_0^{2^{n(1-\varepsilon)}}$.

In view of the large number of sets in $\mathscr{X}_n$, one would expect a wide variety of behavior for its elements. For example, the second author has shown [Gr (63) b] that for all $m \geqslant 78$, there is a set $\{ x_1, \ldots, x_t \} \in \mathscr{X}$ with $\sum_{k=1}^{t} x_k = m$. Furthermore, this is not possible for $m = 77$. It seems highly likely that for any polynomial $p : \mathbf{Z} \to \mathbf{Z}$ it is true that for all sufficiently large m, there is a set $\{ x_1, \ldots, x_t \} \in \mathscr{X}$ with $\sum_{k=1}^{t} p(x_k) = m$, provided p satisfies the obvious necessary conditions:

(i) The leading coefficient of p is positive;

(ii) $gcd\big(p(1), p(2), \ldots\big) = 1$.

It is known [Cas (60)] that conditions (i) and (ii) are sufficient for expressing every sufficiently large integer as a sum $\sum_{a_i \text{ distinct}} p(a_i)$. It has been shown by Burr [Burr (∞)] that for any k, every sufficiently large integer occurs as a sum $\sum_i x_i^k$ for some $(x_1, \ldots, x_m) \in \mathscr{X}'$.

It is trivial to see that $\min \{ x_1 : (x_1, \ldots, x_n) \in \mathscr{X}'_n \} = n$. Since

$$\sum_{u \leq k \leq eu} 1/k = 1 + o(1)$$

then the corresponding quantity

$$\min \{ x_1 : \{ x_1, \ldots, x_n \} \in \mathscr{X}_n \} = f(n)$$

satisfies

$$f(n) \geqslant (1 + o(1)) \frac{n}{e-1} .$$

As far as we know

$$f(n) = (1 + o(1)) \frac{n}{e-1}$$

could hold. In the same way, we see that

$$\min \{ x_n : \{ x_1, \ldots, x_n \{ \in \mathscr{X}_n \} \geqslant (1 + o(1)) \frac{e}{e-1} n$$

and, as before, it may be that equality also holds here. It follows from our previous remarks that

$$\max \{ x_n : \{ x_1, \ldots, x_n \} \in \mathscr{X}_n \} = u_n$$

for $n \geqslant 3$. In general one could ask for

$$\max \{ x_k : \{ x_1, \ldots, x_n \} \in \mathscr{X}_n \} \text{ for a given } k = k(n) .$$

(For $\mathscr{X}'_n$ this turns out to be very easy — just use the "greedy" algorithm up to x_{k-1} and choose the remaining x_j's to be equal).

Is it true that

$$\min \{x_n - x_1 : \{x_1, \ldots, x_n\} \in \mathscr{X}_n\} = (e-1)\, n + o(n)?$$

It is not hard to show that it is greater than $(e-1)\, n + \frac{ng(n)}{\log n}$ for some function $g(n) \to \infty$. It might already be hard to prove this for $g(n) > (\log n)^{\varepsilon}$.

It is well known that for $\{ x_1, \ldots, x_n \} \in \mathscr{X}$, $\max (x_{k+1} - x_k) > 1$, i.e., the sum of the reciprocals of consecutive integers can never be 1, and in fact, can never be an integer (e.g., see [Th (15)], [Kü (18)], [Er (32)]). Is it true that $\max (x_{k+1} - x_k) \geqslant 3$? The decomposition $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$ shows that equality can occur here but we do not know if it can occur infinitely often (or even ever again). It follows from a special case of hypothesis H (a plausible but hopeless conjecture for primes [Sch-Sie (58)]), namely that between x and $2x$ we eventually always have k consecutive integers of the form $q_1, 2q_2, 3q_3, \ldots, kq_k$ where the q_i are primes, that

$$\max (x_{j+1} - x_j) \leqslant k$$

can hold for only finitely many $\{x_1, ..., x_n\} \in \mathscr{X}$. It is not hard to see that there is a function $f(n)$ tending to infinity with n so that the sum $\frac{a}{b}$ of the reciprocals of n consecutive integers always has $b \geqslant f(n)$. In fact it is not too difficult to show that the correct order of growth of $f(n)$ is $e^{n+o(n)}$ although its exact value seems hopeless. The same type of result also holds if the denominators form an arithmetic progression. It would be interesting to know which n consecutive integers minimize b. It can be shown that the largest must be $n + o(n)$ although it is not always n.

Suppose we let $\sum_{a,b}$ denote the sum $\sum_{i=0}^{b-a} \frac{1}{a+i}$. It is probably true that $\sum_{a,b} + \sum_{c,d}$ is an integer only finitely often. However this will probably be difficult to prove since we can not even show that $\sum_{a,b} + \frac{1}{n}$ is an integer only a finite number of times. Perhaps for each k, $\sum_{i=1}^{k} \sum_{a_i,b_i}$ can be an integer only finitely often. It seems likely that for large k, we can always write $1 = \sum_{i=1}^{k} \sum_{a_i,b_i}$ with $b_i > a_i$ (e.g., see [Hah (78)]). An example [Mon (79)] of such a representation for 2 is given by taking the denominators $\{2, 3, 4, 5, 6, 7, 9, 10, 17, 18, 34, 35, 84, 85\}$. It is not hard to show that $\sum_{a,b} = \frac{1}{n}$ is only possible if $b = a = n$. In fact, if $\sum_{a,b} = \frac{u_{a,b}}{v_{a,b}}$ and $b > a$ then $v_{a,b} \geqslant a(a+1)$. In general, $v_{a,b}$ is increasing with b but there can be breaks in the increase (e.g., $\sum_{3,5} = \frac{37}{60}$, $\sum_{3,6} = \frac{19}{20}$). For fixed a what is the least $b = b(a)$ such that $v_{a,b+1} < v_{a,b}$? In fact, is there always such a b for every a?

If we set $\sum_{k=1}^{n} 1/k = \frac{a}{L_n}$ where $L_n = \text{lcm}\{1, ..., n\}$ then is it true that infinitely often we have $(a, L_n) = 1$ and infinitely often we have $(a, L_n) > 1$? It seems likely that

$$\lim_{n\to\infty} \min \left\{ \frac{x_n}{x_1} : \{x_1, ..., x_n\} \in \mathscr{X}_n \right\} = e.$$

For n fixed this is a finite problem—some numerical results might be of interest here for small values of n. It seems obvious that $\min\{x_n/x_1\} = o(\log n)$, for example, but we do not see how to do even this.

What are the possible values of x_n as $\{x_1, ..., x_n\}$ ranges over $\mathscr{X}$? As noted by Straus, the set of x_n is closed under multiplication. Is it true that x_n assumes almost all integer values? Note that x_n is never a prime power, in fact $x_n \neq ap^k$ if p is a prime exceeding $a\,!\log a$. What are the x_n with no element exceeding 1 in $\{x_n\}$ as a proper divisor? Which x_n are not products of two (or more) elements of $\{x_n\}$? How many integers $x_i \leqslant n$ can occur as a element of $\{x_1, ..., x_m\} \in \mathscr{X}$? Are there $o(n)$, cn or $n - o(n)$ such integers?

What is the least integer $v(n) > 1$ which does not occur as an x_k, k variable, for $\{x_1, ..., x_n\} \in \mathscr{X}_n$? It is easy to see that $v(n) > cn\,!$ using results of Bleicher and Erdös [Bl-Er (75)], [Bl-Er (76) a], [Bl-Er (76) b]. It may be that $v(n)$ actually grows more like $2^{2^{\sqrt{n}}}$ or $2^{2^{n(1-\varepsilon)}}$. The corresponding question for $\mathscr{X}'_n$ is also interesting here.

Denote by $k_r(n)$ the least integer which does not occur as x_r in any $\{x_1, ..., x_t\} \in \mathscr{X}$ with $x_1 < ... < x_t \leqslant n$. It is easy to show

$$k_1(n) < \frac{cn \log \log n}{\log n}$$

and with a slight refinement we can get

$$k_1(n) < \frac{cn}{\log n}.$$

We have no idea of the true value of $k_r(n)$ or even of $k_1(n)$.

Suppose we define $K(n)$ to be the least integer which does not occur as x_i for any i in any $\{x_1, ..., x_t\} \in \mathscr{X}$ with $x_1 < ... < x_t \leqslant n$. Again

$$K(n) < \frac{cn}{\log n}$$

is easy but at present we do not even know if $k_1(n) < K(n)$.

Let U_n denote

$$\min \{k : \{x_1, ..., x_k\} \in \mathscr{X},\ x_1 \geqslant n\}.$$

It seems likely that

$$\lim_n \{U_n - (e-1)n\} = \infty$$

but we cannot prove this. It was shown by Erdös and Straus [Er-Str (71) a] that

$$(e-1)n - c < U_n < (e-1)n + c'n/\log n.$$

How many disjoint sets $S_i \in \mathscr{X}$, $1 \leqslant i \leqslant k$, can we find so that $S_i \subseteq \{1, 2, ..., n\}$? No doubt $k = o(\log n)$ but we have not proved this. More generally, how many disjoint sets $T_i \subseteq \{1, 2, ..., n\}$ are there so that all the sums $\sum_{t \in T_i} \frac{1}{t}$ are equal. By using strong Δ-systems [Er-Ra (60)], it can be shown that there are at least $n/e^{c\sqrt{\log n}}$ such T_i. Is this the right order of magnitude? One can also ask how many disjoint sets $\{x_1, ..., x_k\} \in \mathscr{X}_k$ are possible. It is trivial to show that there are no more than $\log k + O(\log \log k)$; however, it is probably true that there are only $o(\log k)$ such sets. Suppose we drop the restriction of disjointness and ask for the number of subsets $S \subseteq \{1, 2, ..., n\}$ which belong to $\mathscr{X}$. Are there 2^{cn} such subsets? $2^{n-o(n)}$? We can show that if $f(n)$ denotes the maximum number of subsets $T \subseteq \{1, 2, ..., n\}$ which all have the same sums $\sum_{t \in T} \frac{1}{t}$ then for any k,

$$n\left(1 - \frac{\log_k n}{\log n} \prod_{i=3}^{k} \log_i n\right) < \frac{\log f(n)}{\log 2} < n\left(1 - \frac{1}{\log n} \prod_{i=3}^{k} \log_i n\right)$$

where $\log_i x$ denotes the i-fold iterated logarithm of x.

Suppose we arbitrarily split the integers into r classes. Is it true that some element of $\mathscr{X}$ belongs entirely to one class? A stronger conjecture is that any sequence $x_1 < x_2 < ...$ of positive density contains a subset $x \in \mathscr{X}$. This is not true if we just assume $\sum_k 1/x_k = \infty$ as the set of primes shows. However, perhaps $\sum_{k=1}^{n} 1/x_k$ cannot grow much faster than this (i.e., $\log \log n$) for the x_i's to fail to contain an $\bar{x} \in \mathscr{X}$.

For a given k, let $a_1 < a_2 < ... < a_t$ satisfy $a_{i+1} - a_i \leqslant k$ and suppose no sum $\sum_{i=1}^{t} \frac{\varepsilon_i}{a_i}$, $\varepsilon_i = 0$ or 1, is equal to 1. Probably a_t is bounded in terms of a_1 and k but we have not excluded the possibility that an infinite sequence with this property exists.

Let $A(n)$ denote the largest value of $|S|$ such that $S \subseteq \{1, 2, ..., n\}$ contains no set in $\mathscr{X}$. Probably $A(n) = n + o(n)$ but we cannot prove this. A related question is to estimate the number of solutions of $1 = \sum_{i=1}^{n} \frac{1}{x_i}$, where $\varepsilon n < x_1 < ... < x_n$.

What is the smallest set $S' \subseteq \{1, 2, ..., n\}$ which contains no set in $\mathscr{X}$ and which is maximal in this respect? We have no idea about this. More generally one could ask for the largest subset S_n^* of $\{1, 2, ..., n\}$ so that for any elements $s, s_1, ..., s_m \in S_n^*$, $\frac{1}{s} \neq \sum_{k=1}^{m} \frac{1}{s_k}$ where $m > 1$. We can certainly have $|S_n^*| > cn$ as the set $\left\{i : \frac{n}{2} < i \leqslant n\right\}$ shows. Can $|S_n^*| > cn$ for $c > \frac{1}{2}$? Szemerédi just asks: Suppose $S_n \subseteq \{1, 2, ..., n\}$ so that $\sum_{s \in S_n} \frac{\varepsilon_s}{s} = \frac{1}{t}$, $\varepsilon_s = 0$ or 1 implies $\sum_s \varepsilon_s = 1$. Perhaps $|S_n| > \varepsilon n$ is no longer possible here. In fact, is it true that if $S \subseteq \{1, 2, ..., n\}$ with $|S| > cn$ then S contains t, x and y with $\frac{1}{t} = \frac{1}{x} + \frac{1}{y}$ Of course, $\frac{1}{t} = \frac{1}{x} + \frac{1}{y}$ holds if and only if $x + y \mid xy$. Suppose $X \subseteq \{1, 2, ..., n\}$ so that $x, y \in X$ implies $x + y \nmid xy$. Can X be substantially more than the odd numbers? What if $x, y \in X$, $x \neq y$, implies $x + y \nmid 2xy$? Must we have $|X| = o(n)$ in this case?

One can ask questions concerning the regularity (in the sense of Rado [Rad (33) b]) of systems of equations involving unit fractions, e.g., is it true that if the integers are split into r classes, some class contains x, y and z satisfying $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$? Or is it true that one class always contains integers whose reciprocals sum to each rational $\frac{a}{b}$?

Let $N(a, b)$ denote the least t for which $\frac{a}{b} = \sum_{k=1}^{t} \frac{1}{x_k}$, $x_1 < x_2 < ... < x_t$, is possible. Erdös [Er (50) b] proved

$$c \log \log b < N(b) = \max_{1 \leq a \leq b} N(a, b) < \frac{c' \log b}{\log \log b}$$

improving the earlier unpublished upper bound of $\frac{c' \log b}{\log \log \log b}$ of de Bruijn. The true order of $N(b)$ seems very hard to determine. Even showing that $N(b) = o\left(\frac{\log b}{\log \log b}\right)$ would be of interest. The upper bound on $N(b)$ can be deduced from the following.

Lemma. Every number less than n ! is the sum of fewer than n distinct divisors of n !

No doubt very many fewer than n divisors are required when n is large, perhaps even only $(\log n)^c$, which would then imply $N(b) < c' \log\log b$.

Denote by n_k the largest integer for which every $m \leqslant n_k$ is the sum of k or fewer distinct divisors of n_k. It would be of interest to estimate n_k —numerical results would also be of interest here.

Let $D(a, b) = \min \max \left\{ x_i : \frac{a}{b} = \sum_{i=1}^{t} \frac{1}{x_i} \right\}$ where the minimum ranges over all decompositions of $\frac{a}{b}$ into a sum of distinct unit fractions, and let $D(b) = \max_{0<a<b} D(a, b)$. Bleicher and Erdös [Bl-Er (76) a] have shown that

$$D(b) < cb(\log b)^2$$

and, in the other direction, if b is a prime p then

$$D(p) \geqslant c'p \log p .$$

It is conjectured that for every $\varepsilon > 0$,

$$D(b) \leqslant c(\varepsilon)\, b\, (\log b)^{1+\varepsilon} .$$

One can also investigate the related quantity

$$n(b) = \frac{1}{b} \sum_{a=1}^{b} N(a, b) .$$

It is known here that $n(b) > c \log\log b$.

The authors (unpublished) have proved that for any $\frac{a}{b}$ with b squarefree there are infinitely many disjoint sets $S = \{ s_1, ..., s_r \}$ such that each s_k is a product of three distinct prime factors and $\frac{a}{b} = \sum_{i=1}^{r} \frac{1}{s_k}$. Whether this can be done with two prime factors is not clear.

In [Bar (77)], Barbeau gives an example of $\{ x_1, ..., x_{101} \} \in \mathscr{X}$ with each x_i the product of two distinct primes. Earlier Burshtein [Burs (73)] gave an example of $\{ x_1, ..., x_n \} \in \mathscr{X}$ with $x_i \nmid x_j$ for $i < j$. However, as Barbeau notes [Bar (76)], it is not known if 1 can be expressed as the product of two sums of the form $\frac{1}{q_1} + ... + \frac{1}{q_k}$ where the q_i are distinct primes. Perhaps this can be done if the q_i are just assumed to be pairwise relatively prime.

Consider the set S_n of all integers which can be written in the form $\sum_{k=1}^{r} \frac{1}{x_k}$ with $1 \leqslant x_1 < \ldots < x_r \leqslant n$, r variable. What is the smallest integer not in S_n? Is it true that $m \notin S_n$ implies $m + 1 \notin S_n$? There are certainly n element sets Y_n such that the sums $\sum_{k=1}^{r} \frac{1}{y_k}$, $y_i \in Y_n$, r variable, represent more integers than can be represented by taking $Y_n = \{1, 2, \ldots, n\}$. To see this, let E_n' be the integer defined by

$$\sum_{k=1}^{n}{}' \frac{1}{k} < E_n' \leqslant \sum_{k=1}^{n+1}{}' \frac{1}{k}$$

where a primed sum indicates that the multiples of primes exceeding $n/\log n$ have been omitted. (Clearly such denominators cannot be used in any representation of an integer). Now adjoin $b_1, \ldots, b_r$ used in a shortest representation of

$$E_n' - \sum_{k=1}^{n}{}' \frac{1}{k} = \frac{1}{b_1} + \ldots + \frac{1}{b_r}.$$

By the previously mentioned estimates in [Er (50) b], r will be less than the number of terms $\leqslant n$ omitted by the primed sums and so, we have a set of fewer than n numbers whose reciprocal sums represent more integers than the reciprocal sums of $1, 2, \ldots, n$.

In fact, if we write

$$\sum_{k=1}^{n} \frac{1}{k} = E_n + t$$

where $0 \leqslant t < 1$ and E_n is an integer, then if t is not too small (as a function of n), we can find n integers $1 \leqslant a_1 < \ldots < a_n$ for which the sums $\sum_{k=1}^{a} \frac{\varepsilon_k}{a_k}$, $\varepsilon_k = 0$ or 1, represent all the integers $1, 2, \ldots, E_n$. To see this, start with $a_1 = 1$. In general, if $a_1, \ldots, a_m$ have been defined for some $m \leqslant 1$, define d by

$$d^* = \sum_{i=0}^{d}{}^* \frac{1}{x_m + i} \leqslant 1 < \sum_{i=0}^{d+1}{}^* \frac{1}{x_m + i}$$

where x_m is the least integer not occurring in $a_1, \ldots, a_m$ and the * indicates that no denominator in the sum can be an a_k. Now, represent d^* as economically as possible, using the result that $\frac{a}{b}$ can be represented as a sum of at most $\dfrac{c \log b}{\log \log b}$ unit fractions if $a \leqslant b$. We then adjoin all these new denominators to the a_k sequence. By continuing this process, we can form $a_1, \ldots, a_u$, $u \leqslant n$, so that if $t = t(n)$ is not too small, then all integers $1, 2, \ldots, E_n$, can be represented by sums of reciprocals of the a_k.

We have no idea of how many integers can be written as $\sum_{k=1}^{n} \dfrac{\varepsilon_k}{k}$, $\varepsilon_k = 0$ or 1. We cannot even rule out the possibility that there are more than $c \log n$ integers of this form.

For a fixed $c > 0$, suppose $S_c \subseteq \{1, 2, \ldots, n\}$ with $|S_c| \geqslant cn$. Is it true that there is a function $f(c)$ so that some sum $\sum_{s \in S_c} \dfrac{1}{s} = \dfrac{a}{b}$ has $b \leqslant f(c)$?

What is $\min_{S_n} \left| 1 - \sum_{s \in S_n} \dfrac{1}{s} \right|$ where $S_n \subseteq \{1, 2, \ldots, n\}$ ranges over all sets containing no set of $\mathscr{X}$, i.e., $1 \neq \sum_{s \in S_n} \dfrac{\varepsilon_s}{s}$, $\varepsilon_s = 0$ or 1. It should be $e^{-(c+o(1))n}$ for some c, $0 < c < 1$. It is trivially at least $\operatorname{lcm}(1, 2, \ldots, n)^{-1}$ and it is probably much larger.

Is it true that there is a $c > 0$ such that for fixed α and t sufficiently large, if $\sum_{k=1}^{t} \dfrac{1}{s_k} > \alpha$ then for some choice of $\varepsilon_k = 0$ or 1,

$$0 \leqslant 1 - \sum_{k=1}^{t} \frac{\varepsilon_k}{s_k} < e^{-c\alpha}\,?$$

We know only c/α^2 as an upper bound at present.

Although $\dfrac{1}{2} + \dfrac{1}{3} = 1 - \dfrac{1}{6}$ and $\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} = 1 + \dfrac{1}{12}$, probably

$$\left| \sum_{k=2}^{n} \frac{1}{k} - I \right| > \frac{1}{L_n}$$

for every other n where I is an integer and $L_n = \operatorname{lcm}(1, 2, \ldots, n)$. Can we have $\dfrac{1}{q_1} + \ldots + \dfrac{1}{q_t} + \dfrac{1}{m} = 1$ infinitely often where $q_1, \ldots, q_t$ are distinct primes, such as $\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{6} = 1$? It is not difficult to give solutions to

$$\frac{1}{a_1} + \ldots + \frac{1}{a_n} + \frac{1}{\operatorname{lcm}(a_1, \ldots, a_n)} = 1\,.$$

Choose $t = t(n)$ to be the least integer such that $\varepsilon_n = \sum_{k=n}^{t} \frac{1}{k} - 1 \geqslant 0$. How small can ε_n be? As far as we know this has not been looked at. It should be true that $\liminf_n n^2 \varepsilon_n = 0$ but perhaps $n^{2+\delta}\varepsilon_n \to \infty$ for every $\delta > 0$. The quantity $t\varepsilon_n$ is equidistributed modulo 1 and, in fact, is probably uniformly distributed. In any case, it might be of interest to obtain results on its distribution function. Similar questions can be asked for $n \left| I - \sum_{k=2}^{n} \frac{1}{k} \right|$ where I is an integer and $\sum_{k=2}^{n} \frac{1}{k} \leqslant I < \sum_{k=2}^{n+1} \frac{1}{k}$.

With u_n defined as before, i.e., $u_1 = 1$, $u_{n+1} = u_n(u_n+1)$, we have $\sum_{k=1}^{\infty} \frac{1}{u_{k+1}} = 1$ and $u_k = [c_0^{2k}+1]$, $k \geqslant 1$, where $c_0 = 1.264085...$. If $a_1 < a_2 < ...$ is any other sequence with $\sum_{k=1}^{\infty} \frac{1}{a_k} = 1$ is it true that

$$\liminf_n a_n^{1/2^n} < \lim_n u_n^{1/2^n} = c_0 \ ?$$

Is it true that if $a_1 < a_2 < ... < a_t$ and $\sum_{k=1}^{t} \frac{1}{a_k} < 2$ then there exist $\varepsilon_k = 0$ or 1 so that

$$\sum_{k=1}^{t} \frac{\varepsilon_k}{a_k} < 1 \text{ and } \sum_{k=1}^{t} \frac{1-\varepsilon_k}{a_k} < 1 \ ?$$

This is not true if we just assume $a_1 \leqslant a_2 \leqslant ... \leqslant a_t$, as, for example, the sequence 2, 3, 3, 5, 5, 5, 5, shows. It is conjectured by Spencer and the authors that in any case, if $\sum_{k=1}^{t} \frac{1}{a_k} \leqslant N - \frac{1}{30}$ then the a_k can be split into N sequences $a_k^{(i)}$, $1 \leqslant i \leqslant N$, so that $\sum_k \frac{1}{a_k^{(i)}} \leqslant 1$ for all i.

Recently, in [Wo (76)] the quantity

$$r(n) = \min_{\varepsilon_i} \left\{ \left| \alpha - \sum_{i=1}^{n} \frac{\varepsilon_i}{i} \right| : \varepsilon_i = 0 \text{ or } 1 \right\}$$

is investigated. It is shown there that $r(n)$ is at most $n^{-c_\alpha \log n}$. No doubt much more is true if $\alpha \in (0, 1)$. It is certain that

$$r(n) < e^{-(\log n)^k}$$

for all k if $n \geqslant n_0(k)$ and in fact it is probably true that $r(n) < e^{-n^{\varepsilon}}$ for some $\varepsilon > 0$.

These questions lead to the consideration of the distribution of the sums $\sum_{k=0}^{n} \frac{\delta_k}{k}$ where $\delta_k = 0$ or ± 1. It should be easy to see that there is a $c > 0$ so that the inequality

$$\min_{\delta_k} \left\{ \left| \sum_{k=1}^{n} \frac{\delta_k}{k} \right| \right\} < \frac{c}{2^n}, \ \delta_k = 0, \ \pm 1$$

holds for all n where the value 0 is not allowed. Unfortunately, we do not see how to prove this at present. It seems quite likely that there is a $c > 0$ independent of n so that

$$\lim_{n} \ (2+c)^n \min_{\delta_k} \sum_{k=1}^{n} \frac{\delta_k}{k} = 0 \, .$$

Of course,

$$\left| \sum_{k=1}^{n} \frac{\delta_k}{k} \right| \geqslant \frac{1}{L_n}$$

where $L_n = \operatorname{lcm} \{ 2, 3, \ldots, n \}$. For large n we no doubt must have inequality but this we cannot prove. Examples of equality exist for small n, e.g.,

$$\frac{1}{2} - \frac{1}{3} - \frac{1}{4} = - \frac{1}{12} \, .$$

Erdös and Straus [Er-Str (75)] showed that for any nonconstant sequence δ_k, $k = 1, 2, \ldots$, of $\pm$ 1's there is a finite subsequence for which $\sum_k \frac{\delta_{i_k}}{i_k} = 0$. R. Sattler [Sat (75)] proved the corresponding more difficult result for $\sum_k \frac{\delta_{i_k}}{2i_k + 1}$. Is this also true for the general case $\sum_k \frac{\delta_{i_k}}{a i_k + b}$? What about for any set of denominators of positive density? Of course, this cannot hold for all choices of the δ_k for the case $\sum_k \frac{\delta_{i_k}}{i_k^2}$ since $\sum_{k \leq 2} \frac{1}{k^2} < 1$. However, it is conceivable that it is still true if we restrict k to be at least 2, i.e., for any nonconstant sequence $\delta_2, \delta_3, \ldots$ of $\pm$ 1's, there is a finite subsequence δ_{i_k} for which $\sum_k \frac{\delta_{i_k}}{i_k^2} = 0$.

How large can a set $A \subseteq \{1, 2, ..., n\}$ be so that for some choice of $\delta_k = \pm 1$, $a \in A$, we have:

(i) $\sum_{a \in A} \frac{\delta_a}{a} = 0$;

(ii) For every nonempty proper subset $A' \subset A$,

$$\sum_{a \in A'} \frac{\delta_a}{a} \neq 0 ?$$

A question which has received some attention in the literature is the following: What is the number $t(n)$ of distinct sums of the form $\sum_{k=1}^{n} \frac{\varepsilon_k}{k}$, $\varepsilon_k = 0$ or 1? The best estimates [Bl-Er (75)] for $t(n)$ are

$$\frac{n}{\log n} \prod_{i=3}^{k} \log_i n \leqslant \frac{\log t(n)}{\log 2} < \frac{n \log_k n}{\log n} \prod_{i=3}^{k} \log_i n$$

for $k \geqslant 4$ and $\log_k n \geqslant k$. A related question is the following. How many integers $a_1 < a_2 < ... < a_{r(n)} \leqslant n$ can we have so that all the sums $\sum_{i=1}^{r(n)} \frac{\varepsilon_i}{a_i}$ are distinct? Estimates of Bleicher and Erdös [Bl-Er (75)] imply that

$$\frac{n}{\log n} \prod_{i=3}^{s} \log_i n < r(n) < \frac{n \log_s n}{\log n} \prod_{i=3}^{s} \log_i n$$

for any fixed s. Is it true that $\frac{\log t(n)}{r(n)} \to \infty$ with n? Here one can also ask for a maximal such set of a_i's having as few elements as possible. It is easy to see that the number of elements in such a set has a greater order of magnitude than $\pi(n)$. We don't know whether this is the case if instead we require all products $\prod_{i=1}^{r(n)} \frac{1}{a_i^{\varepsilon_i}}$ to be distinct.

Let $a_1 < a_2 < ...$ be an infinite sequence of integers and let $f(n)$ denote the number of solutions of

$$1 = \sum_{k=1}^{n} \frac{\varepsilon_k}{a_k}, \quad \varepsilon_k = 0 \text{ or } 1 .$$

It is possible to have

$$\lim_n f(n)^{1/n} = 2$$

and it is not hard to prove that for the number $A(n)$ of sums $\sum_{k=1}^{n} \frac{\varepsilon_k}{k}$ which are less than 1, $\lim_n A(n)^{1/n}$ exists and is strictly less than 2. It would be interesting to know the exact value of the limit. If $g(x)$ is any function which tends to infinity with x then for the sums $\sum_{k=1}^{n} \frac{\varepsilon_k}{kg(k)}$ less than 1, the corresponding counting function $A'(n)$ satisfies $\lim_n A'(n)^{1/n} = 2$. Similarly, for $\sum_{k=1}^{n} \frac{\varepsilon_k g(k)}{k}$, the corresponding counting function $A''(n)$ satisfies $\lim_n A''(n)^{1/n} = 1$.

An old conjecture of Erdös and Straus asserts that for all $n > 1$, the equation

$$(*) \qquad \frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

has integer solutions. This has still not been settled. However, Vaughan [Va (70)] and Webb [Web (70)] have each given estimates for the number $f(N)$ of $n \leqslant N$ for which (*) is not solvable. From these we know

$$f(N) < N \exp\{-c(\log N)^{2/3}\}$$

for some $c > 0$. The equation (*) is known to hold for $n \leqslant 10^8$ (see [Franc (78)], [Ter (71)], [Ya (64)], [Ya (65)]).

More generally, it has been conjectured by Schinzel and Sierpiński [Sie (56)] that the equation

$$\frac{a}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

is always solvable for $n > n_0(a)$. Schinzel also conjectured that for every $a \geqslant 1$ there is an $n(a)$ such that all fractions $\frac{a}{n}$ with $n \geqslant n(a)$ can be written in the form

$$\frac{a}{n} = \frac{1}{x} \pm \frac{1}{y} \pm \frac{1}{z}$$

with x, y and z positive integers. At present this has been proved for all $a \leqslant 40$ (see [Str-Sub (xx)]) although there is no infinite class of a's for which it is known to hold. (Also see [Franc (78)], [Pala (58)], [Pala (59)], [Stew$_1$ (64)], [Stew$_1$-We (66)], [Vi (72)], [Web (74)], for related results).

For a rather complete bibliography on unit fractions, the reader should consult [Cam (xx)].

5. Bases and related topics

A sequence $A = (a_1, a_2, ...)$ of integers is called a *basis of order* k if every (positive) integer is the sum of at most k of the a_i's where repetition is allowed. For example, as we have remarked earlier, it is well known that the squares form a basis of order 4. Some of the most famous and intractable problems of number theory deal with bases, e.g., Goldbach's conjecture which states that every even number exceeding 2 is the sum of two primes. The sharpest known results for this conjecture assert that every large odd integer is the sum of three primes and that every large even integer is of the form $p + \theta$ where p is prime and θ has at most two prime factors.

Another well known problem concerned with bases is Waring's problem. J. A. Euler conjectured that every integer is the sum of at most $g(k) = 2^k + [(3/2)^k - 2]$ k^{th} powers. This has been proved for most values of k and is no doubt always true. Hardy and Littlewood introduced the quantity $G(k)$, defined to be the smallest integer so that every *large* number is the sum of at most $G(k)$ k^{th} powers. It is now known that $G(k) < ck \log k$. The truth probably is $G(k) \leqslant 4k$ with equality if k is a power of 2. Wieferich proved $g(3) = 9$ and Landau first showed that $G(3) \leqslant 8$. Dickson proved that 23 and 239 are the only integers which require 9 cubes in their representations. Linnik established $G(3) \leqslant 7$; Watson subsequently obtained a completely different proof which is much simpler. Probably $G(3) = 4$ but it is not even known at present if every large integer can be expressed as a sum $x_1^3 \pm x_2^3 \pm x_3^3 \pm x_4^3$. (See [Ellis (71)] for a survey of results on Waring's problem).

Denote by $f_{k,l}(x)$ the number of integers not exceeding x which are the sum of l nonnegative k^{th} powers. The estimation of $G(k)$ would be greatly improved if we could prove

$$f_{k,k}(x) > x^{1-\varepsilon},$$

or even

$$f_{k,m}(x) > cx^{m/k}$$

for every $m < k$ and $\varepsilon > 0$ and every sufficiently large x. This inequality seems unattackable by the methods at our disposal although the case $k = 2$ is quite easy—actually Landau proved

$$f_{2,2}(x) \sim \frac{cx}{\sqrt{\log x}}.$$

For $k > 2$, it is unknown if $f_{k,k}(x) = o(x)$ and in fact no one has even a well motivated conjecture here.

Denote by $r_{k,l}(n)$ the number of solutions of $n = \sum_{i=1}^{l} x_i^k$. The famous K-hypothesis of Hardy and Littlewood stated

$$r_{k,k}(n) = O(n^{\varepsilon})$$

for every $\varepsilon > 0$. This is easy for $k = 2$. However, Mahler [Mah (36)] disproved the conjecture for $k = 3$. He showed

$$r_{3,3}(n) > cn^{1/12},$$

for large n, which perhaps is the right order of growth (though nothing is known about this). Probably the K-hypothesis fails for every $k > 3$ as well. Hardy and Littlewood also made the following weaker conjecture:

$$\sum_{n=1}^{x} r_{k,k}(n)^2 < x^{1+\varepsilon}$$

for every $\varepsilon > 0$. This is probably true but no doubt very deep. However, it would suffice for most applications.

Mordell proved $\limsup r_{3,2}(n) = \infty$ and Mahler showed $r_{3,2}(n) > (\log n)^{\alpha}$ for infinitely many n and some $\alpha > 0$. It is also known that $\limsup r_{4,2}(n) \geqslant 2$ and $\limsup r_{4,3}(n) = \infty$ (see [Lag (75)]); beyond this, nothing much is known. For example, it is not known if $x^5 + y^5 = u^5 + v^5$ has any nontrivial solutions. Euler conjectured that

$$x^n = \sum_{i=1}^{n-1} y_i^n$$

has no nontrivial solutions. This is well known for $n = 3$, unknown for $n = 4$ and was disproved [Lan-Pa (66)] for $n = 5$:

$$144^5 = 133^5 + 110^5 + 84^5 + 27^5 .$$

It was proved by Mahler and Erdös [Er-Ma (38)] that the number of distinct integers not exceeding n of the form $x^k + y^k$, $k > 2$, is greater than $cn^{2/k}$ (in fact, they prove a more general theorem). Hooley [Hoo (64)] strengthened this result by obtaining an asymptotic formula. It would be very interesting to prove that the number of integers less than n of the form $x_1^k + x_2^k + x_3^k$, $k \geqslant 3$, exceeds $cn^{3/k}$ or even $n^{3/k-\varepsilon}$. This would be very useful even for large k but at present only much weaker results are known.

For $k = 3$ the sharpest inequality is an old result of Davenport [Dave (50)]:

$$f_{3,3}(n) > n^{\frac{47}{54} - \varepsilon}.$$

Denote by $f_k(n)$ the number of solutions of

$$n = \sum_{i=1}^{k} x_i^k.$$

It was shown by Erdös [Er (36) b] (and independently by S. Chowla) that

$$f_k(n) > n^{c/\log \log n}.$$

If the x_i are restricted to be primes then the corresponding number $f'_n(n)$ has been studied in [Er (37)]. It is known that

$$f'_2(n) > n^{c/\log \log n}$$

and it can be shown that $\max_{1 \leq n \leq x} f_3(n)$ tends to infinity with x fairly rapidly. However, for $k \geqslant 4$ almost nothing is known. For further problems and results in this direction, see the paper of Erdös and Szemerédi [Er-Sz (72)]. However, as we have stated, we do not wish to discuss the classical questions too much in this paper so we leave this topic now.

Let $a_1 < \ldots < a_k \leqslant x$ be such that every $n < x$ is of the form $a_i + a_j$. Rohrbach [Roh (37)] conjectured that under this hypothesis

$$k > 2\sqrt{x} + O(1).$$

However, Rohrbach's conjecture has recently been disproved by Hämmerer and Hofmeister [Häm-Hofm (76)].

It was asked by Erdös whether there is an infinite sequence $\{ a_k \}$ for which $n = a_i + a_j$ is solvable for every n and which satisfies $a_k/k^2 \to c$. Cassels [Cas (57)] gave an example of such a basis which in fact satisfies

$$a_k = ck^2 + O(k).$$

Nevertheless, there is a small amount of "cheating" going on here. Cassels actually gives a basis $\{ b_k \}$ where lim sup b_k/k^2 differs from lim inf b_k/k^2 and then adds new terms. The "correct" way of formulating the question is this: Suppose $a_1 < a_2 < \ldots$ is a basis of order 2 such that the removal of any a_i destroys the basis property, i.e., the a_k's form a minimal basis. Can it happen that

$$\lim_{k \to \infty} a_k/k^2 = c\ ?$$

We conjecture that it cannot. Another way of stating the problem is this: Does every basis of order 2 have a subset $\{ a_k \}$ which is also a basis and for which $\lim_{k \to \infty} a_k/k^2$ does not exist?

For the sequence $A = \{ a_k \}$, let $f(n) = f_A(n)$ denote the number of solutions to $n = a_i + a_j$, $i < j$. We mention several interesting questions concerning $f(n)$ which have been open for some time now.

1. (\$250) Is there a sequence $\{ a_k \}$ for which $\lim_n \dfrac{f(n)}{\log n}$ exists and is greater than 0? It can be shown by probability methods that there exists $\{ a_k \}$ with
$$c_1 \log n < f(n) < c_2 \log n .$$
More specifically, between 2^k and 2^{k+1} choose a random subset of size $c \dfrac{2^{k/2}}{\sqrt{k}}$, for $k \geqslant k_0$. Then almost all such choices satisfy the desired inequalities. It can also be shown by these methods that there exist sequences $\{ a_k \}$ for which $f(n) \sim g(n) \log n$ where $g(n)$ tends to infinity arbitrarily slowly.
2. Give an explicit construction of a sequence $\{ a_k \}$ which has $1 \leqslant f(n) = o(n^\varepsilon)$ for all $\varepsilon > 0$.
3. (Erdös-Turán — \$500) Show that $f(n) > 0$ for all n implies $\limsup f(n) = \infty$. Is it actually true that this implies $f(n) > c \log n$?
4. Show that $a_k < ck^2$ implies $\limsup_n f(n) = \infty$. It is known that there are sequences $\{ a_k \}$ with $\liminf_n a_k/k^2 < \infty$ and $f(n) \leqslant 1$.
5. Ajtai, Komlós and Szemerédi [Aj-Ko-Sz (xx)] have just shown that there exists $\{ a_k \}$ with $a_k = o(k^3)$ and $f(n) \leqslant 1$ for all n. Probably a_k can be chosen so that $a_k < ck^{2+\varepsilon}$. It is known that for all $\varepsilon > 0$ there exists c_ε and a sequence $\{ a_k \}$ with $a_k < k^{2+\varepsilon}$ for $k \geqslant k_0$ and $f(n) < c_\varepsilon$.
6. Suppose $a_n < cn^3$ for all n. Is it true that not all the triple sums $a_i + a_j + a_k$ can be distinct? Bose and Chowla (see Proceedings of the 1959 Boulder Number Theory Conference) proved that it is possible to select $(1 + o(1))\, x^{1/3}$ integers less than x so that all triple sums are distinct. They asked if this is possible for $(1+\varepsilon)\, x^{1/3}$ integers.
7. (P. Erdös and D. J. Newman). Suppose $f_A(n) \leqslant c$ for all n. Is it always possible to partition A into $t = t(c)$ subsets $A_1, \ldots, A_t$ such that $f_{A_k}(n) < c$ for all k and n? J. Nešetřil and V. Rödl (personal communication)

have recently shown that the answer is negative for all c .Erdös had previously shown this for $c = 3, 4$ and infinitely many other values.

8. Suppose $f_A(n) \leqslant 1$ for all n. How large can $p_A = \limsup_n \frac{A(n)}{\sqrt{n}}$ be?

It was shown by Erdös (see [St (55) *]) that $p_A = \frac{1}{2}$ is possible and later by Krückeberg [Krüc (61)] that $p_A = \frac{1}{\sqrt{2}}$ is possible. It can be shown [Er-Tu (41)] that $p_A \leqslant 1$ always holds. Is it true that any finite set with all pair sums distinct can be embedded in some (finite) perfect difference set? (If so then it would follow that $p_A = 1$ is possible).

For other problems and results on this subject, see the recent papers of Erdös and Nathanson [Na (74)], [Er-Na (75) a], [Er-Na (75) b], [Er-Na (76)], [Er-Na (77)], [Na (77) b], [Na (77) c], [Er-Na (78)].

A sequence $a_1 < a_2 < \ldots$ is called an *essential component* if for every sequence $b_1 < b_2 < \ldots$ with Schnirelman density α, the set of all sums $\{ a_i + b_j \}$ has density strictly greater than α. By a result in Wirsing's thesis (also see [Plü (69)]) it follows that in fact the density of $\{ a_i + b_j \}$ is greater than $\alpha + f(\alpha)$ where f depends on $\{ a_i \}$ but not on $\{ b_j \}$ or α. In 1935 it was shown by Erdös [Er (35) b] that every basis is an essential component. However, Linnik [Linn (42)] disproved the converse by giving an example of an essential component $A = \{ a_1, a_2, \ldots \}$ which is not a basis. His example satisfies

$$A(x) < e^{(\log x)^{\frac{9}{10} + \varepsilon}}$$

for every $\varepsilon > 0$ where, as usual, $A(x)$ denotes the number of elements of A which do not exceed x. Linnik's proof is very complicated; Wirsing has recently found a fairly simple proof which appears in [Wir (74)]. His example satisfies

$$A(x) < e^{(\log x)^{\frac{1}{2} + \varepsilon}}$$

for every $\varepsilon > 0$.

We conjecture that if $a_{k+1}/a_k \geqslant c > 1$ then the sequence A cannot be an essential component. Wirsing believes that if

$$A(x) < e^{(\log x)^{\frac{1}{2} - \varepsilon}}$$

for some $\varepsilon > 0$ then A cannot be an essential component; this would settle the question of the order of magnitude.

A question of Erdös and Nathanson is the following. Suppose $a_1 < a_2 < \ldots$ is a minimal basis which has positive density. Can it happen that for any a_k, the (upper) density of the integers which cannot be represented without using a_k is positive?

Let $A = \{ a_1 < a_2 < \ldots \}$ and $B = \{ b_1 < b_2 < \ldots \}$ be sequences of integers satisfying $A(x) > \varepsilon x^{1/2}$, $B(x) > \varepsilon x^{1/2}$ for some $\varepsilon > 0$. Is it true that $a_i - a_j = b_k - b_l$ has infinitely many solutions? Very recently, Prikry, Tijdeman, Stewart, and others (see [Stew$_2$ (xx)], (Ti (xx)]) have shown that if $A = \{ a_1, a_2, \ldots \}$ has positive density, the set of numbers $D(A) = \{ d_1 < d_2 < \ldots \}$ which occur infinitely often as $a_i - a_j$ has bounded gaps, i.e., $d_{i+1} - d_i$ is bounded. In fact, this holds for the intersection $D(A_1) \cap \ldots \cap D(A_n)$ for any n sets A_i of positive density. It would be interesting to know to what extent this conclusion holds for weaker hypotheses on the A_i. We could also ask for the best bound on the gap sizes in terms of the densities. What if we only require that $D(A)$ or $D(A_1) \cap \ldots \cap D(A_n)$ have positive density; or even that $\sum_{d \in D(A)} \frac{1}{d} = \infty$; or even that $D(A) \neq \varnothing$ instead of bounded gaps?

Let A be a set of integers with asymptotic density zero. Does there always exist a basis B with $B(x) = o(\sqrt{x})$ so that every $a \in A$ can be written as $a = b_i + b_j, b_i, b_j \in B$? This is known [Er-Ne (77)] to be possible, for example, when A is the set of squares.

Let $S_n = \{ s_1 < s_2 < \ldots \}$ denote the set of all integers which have all prime factors less than n. What is the largest $m = m(n)$ so that any $t \in [1, m]$ can be written as $t = s_i + s_j$? At present, we don't even see how to show $m \geqslant n^3$. Probably the right answer is about $e^{\sqrt{n}}$.

For a sequence X, let $d(x)$ denote the asymptotic density of X (assuming it exists). Suppose $d(A)$ and $d(B)$ are positive and

$$(*) \qquad d(A+B) = d(A) + d(B)$$

where, as usual, $A + B$ denotes the set $\{ a + b : a \in A, b \in A \}$. One way this can happen is as follows:

Let μ be a probability measure on S^1, the circle of circumference 1, and let $\bar{A}, \bar{B} \subseteq S^1$ with

$$\mu(\bar{A}+\bar{B}) = \mu(\bar{A}) + \mu(\bar{B})$$

where addition on S^1 is just ordinary addition modulo 1. For some $\alpha > 0$, define

$$A = \{ n > 0 : \{ n\alpha \} \in \bar{A} \},$$
$$B = \{ n > 0 : \{ n\alpha \} \in \bar{B} \}.$$

where $\{ x \}$ denotes the fractional part of x.

For example, when μ is Lebesgue measure and $\bar{A}$ and $\bar{B}$ are intervals or when μ is an atomic measure supported on certain rationals and $\bar{A}$ and $\bar{B}$ are certain rationals in S^1 (so that A and B are unions of congruence classes), we can get solutions to (*). Is it true that all solutions are generated in a similar way (using other groups)?

A is called an *asymptotic basis of order r* if every sufficiently large integer is a sum of at most r integers taken from A. We say that A has *exact order t* if every sufficiently large integer is a sum of exactly t integers from A. In the first case we write ord $(A) = r$; in the second case we write ord* $(A) = t$. In a recent paper we show [Er-Gr (79)] that a basis $A = \{ a_1, a_2, \ldots \}$ has an exact order if and only if g.c.d. $\{ a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots \} = 1$. Even when ord* (A) exists it may be larger than ord (A). For example, the set B defined by

$$B = \bigcup_{k=0}^{\infty} I_k$$

where $I_k = \{ x : 2^{2k} + 1 \leqslant x \leqslant 2^{2k+1} \}$ has

$$\text{ord}\,(B) = 2\,,\ \ \text{ord}^*\,(B) = 3\,.$$

However, there are some fairly good bounds known on the extent of this increase. To describe these, define the function $h : \mathbf{Z}^+ \to \mathbf{Z}^+$ as follows:

$$h\,(r) \equiv \max \{ \text{ord}^*\,(A) : \text{ord}\,(A) = r \text{ and } \text{ord}^*\,(A) \text{ exists} \}\,.$$

Then we have shown

$$\frac{1}{4}\,(1 + o\,(1))\,r^2 \leqslant h\,(r) \leqslant \frac{5}{4}\,(1 + o\,(1))\,r^2\,.$$

We have no idea what the right coefficient of r^2 is. It is known that $h\,(2) = 4$. However even $h\,(3)$ is unknown at present (it is $\geqslant 7$).

For a set A, let $A_m\,(x)$ denote $\left| \{ a_{i_1} + \ldots + a_{i_m} : a_{i_k} \in A \} \cap \{ 1, \ldots, x \} \right|$. If A is a basis and $A_1\,(x) = o\,(x)$ is it true that

$$\lim_{x \to \infty} \frac{A_2\,(x)}{A_1\,(x)} = \infty\ ?$$

It has been shown by Freĭman [Fre (73)] that for any sequence B of density zero,

$$\limsup_{x \to \infty} \frac{B_2(x)}{B_1(x)} \geqslant 3 .$$

and that the constant 3 is best possible. The following related result has recently been proved by Ruzsa [Ru (73)]. Let $A_\Delta(x)$ denote $|\{ a_i - a_j \leqslant x : i > j \}|$ for the sequence $A = \{ a_1 < a_2 < ... \}$. Then if A has density zero.

$$\limsup_{x \to \infty} \frac{A_\Delta(x)}{A_1(x)} = \infty .$$

By the *restricted order* of A, denoted by $\mathrm{ord}_R(A)$, we mean the least integer t (if it exists) such that every sufficiently large integer is the sum of at most t *distinct* summands taken from A. As pointed out by Bateman, for $h \geqslant 3$ the set $A_h = \{ 1 \} \cup \{ x > 0 : x \equiv 0 \pmod{h} \}$ has $\mathrm{ord}(A) = h$ but has no restricted order. However, Kelly [Ke (57)] has shown that $\mathrm{ord}(A) = 2$ implies $\mathrm{ord}_R(A) \leqslant 4$ and conjectures that, in fact, $\mathrm{ord}_R(A) \leqslant 3$. What are necessary and sufficient conditions on a basis A to have a restricted order? Is there a function $f(r)$ such that if $\mathrm{ord}(A) = r$ and $\mathrm{ord}_R(A)$ exists then $\mathrm{ord}_R(A) \leqslant f(r)$? What are necessary and sufficient conditions that $\mathrm{ord}(A) = \mathrm{ord}_R(A)$. As we have noted, the situation is not clear even for sequences of polynomial values, e.g., for the set S of squares, $\mathrm{ord}(S) = 4$ and $\mathrm{ord}_R(S) = 5$ (see [Pall (33)]) while the set T of triangular numbers has $\mathrm{ord}(T) = \mathrm{ord}_R(T) = 3$ (see [Sch (54)]).

Is it true that if for some r, $\mathrm{ord}(A-F) = r$ for all finite sets F, then $\mathrm{ord}_R(A)$ exists? What if we just assume $\mathrm{ord}(A-F)$ exists for all finite sets F?

Let $n \times A$ denote the set $\{ a_{i_1} + ... + a_{i_n} : a_{i_k}$ are distinct elements of $A \}$. Is it true that if $\mathrm{ord}(A) = r$ then $r \times A$ has positive (lower) density? If $\{ a_{i_1} + ... + a_{i_s} : a_{i_k} \in A \}$ has positive upper density then must $s \times A$ also have positive upper density?

Of course, many of these questions can be formulated for $\mathrm{ord}_R^*(A)$, defined in the obvious way, but we will not pursue them here.

Let $a_1, a_2, ...$ be the sequence defined by:

(i) $a_1 = 1$;

(ii) For $n \geqslant 1$, a_{n+1} is the least integer exceeding a_n so that all the sums $a_{n+1} + a_k$, $1 \leqslant k \leqslant n$, are distinct from all preceding sums $a_j + a_i$, $1 \leqslant i \leqslant j \leqslant n$.

Thus, the sequence begins 1, 2, 4, 8, 13, 21, 31, 45, 66, 81, 97, ... What is the order of growth of a_n? In particular is $a_n = O(n^{2+\varepsilon})$? Which integers occur as differences $a_j - a_i$? For example, does 22 occur as a difference? Do the differences have positive density? One could also investigate the corresponding question where all triple sums $a_i + a_j + a_k$ are distinct. As far as we know, even the preceding greedy construction yields a sequence with $a_n = O(n^3)$.

The following old problem of Dickson [Dic (34)] is still wide open. Given a set of integers $A_k = \{a_1 < ... < a_k\}$, extend it to the infinite sequence $A = \{a_1 < ... < a_k < ...\}$ by defining a_{n+1} for $n \geqslant k$ to be the least integer exceeding a_n which is *not* of the form $a_i + a_j$, $i, j \leqslant n$. Is it true that the sequence of differences $a_{m+1} - a_m$ is eventually periodic? Even a starting set as small as $\{1, 4, 9, 16, 25\}$ requires thousands of terms before periodicity eventually occurs.

Ulam has raised the following problem. Starting with $a_1 = 1$, $a_2 = 2$, define a_{n+1} for $n \geqslant 2$ to be the least integer exceeding a_n which can be expressed uniquely as $a_i + a_j$, $i \neq j$, $i, j \leqslant n$. Thus, the sequence begins 1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, ... What can be said about this sequence. In particular, do infinitely many pairs a, $a + 2$ occur? Does this sequence (eventually) have periodic differences? Is the density 0? Almost nothing is known (see [Q (72)], [Mia-Ch (49)]).

Finally, we mention the following very annoying problem. Find a polynomial $f(x)$ such that all the sums $f(a) + f(b)$, $a < b$, are distinct. Of course, it is easy to give such polynomials—for example, $f(x) = x^5$ must certainly work. Unfortunately we are not able to prove that this (or any) f actually works.

6. Completeness of sequences and related topics

For a sequence $A = (\alpha_1, \alpha_2, ...)$ of real numbers, let $P(A)$ be defined by

$$P(A) = \left\{ \sum_{k=1}^{\infty} \varepsilon_k \alpha_k : \varepsilon_k = 0 \text{ or } 1 , \sum_{k=1}^{\infty} \varepsilon_k < \infty \right\} .$$

Numerous results are available on the structure of $P(A)$ for various special sequences A, particularly when the a_k are integers or reciprocals of integers (for example, see the survey article [Gr (71)]). The general flavor of questions and results in this area tend to differ from those dealing with *bases* for the integers because the sums under consideration have no restriction on the number of terms used (whereas bases do have this restriction) although the sums are restricted by the multiplicity any particular term can have). For example, for the sequence $A = \{1, 4, 9, 16, \ldots\}$ of squares, it is well known that A is a basis of order 4 (Lagrange's theorem) while it is less well known [Spr (48) a] that $128 \notin P(A)$ and all $m > 128$ belong to $P(A)$, i.e., any integer greater than 128 is a sum of *distinct* squares (but not 128 itself) (also see [Spr (48) b]). In this section, we discuss a variety of what we consider to be interesting problems related to this topic.

Let us call a sequence $S = (s_1, s_2, \ldots)$ of integers *complete* if $P(S)$ contains all sufficiently large integers. We call S *subcomplete* if $P(S)$ contains an infinite arithmetic progression. Finally we call S *strongly complete* if the sequence $(s_n, s_{n+1}, \ldots)$ is complete for every n. For example, the sequence $T = (t_0, t_1, \ldots)$ with $t_n = 2^n$ is complete but not strongly complete, any subsequence $T^{(m)} = (t_m, t_{m+1}, \ldots)$, $m > 0$, is subcomplete but not complete, and the merged sequence $T^* = (t_0, x_0, t_1, x_1, t_2, x_2, \ldots)$ is strongly complete provided infinitely many of the x_i are odd.

One of the nicest open questions on complete sequences is due to Jon Folkman. It is this: If $S = (s_1, s_2, \ldots)$ is a nondecreasing sequence of integers satisfying $s_n < cn$ for some c and all n, then must S be subcomplete? Folkman showed [Fo (66)] that this is true under the stronger hypothesis $s_n < cn^{1-\varepsilon}$ (strengthening earlier results of Erdös [Er (62) a]). On the other hand for all $\varepsilon > 0$, examples can be given (see [Fo (66)], [Cas (60)] of sequences S satisfying $s_n < cn^{1+\varepsilon}$ which are not subcomplete. In [Fo (66)] Folkman also proved the result (conjectured by Erdös) that if the s_n's are strictly increasing and $s_n < cn^{2-\varepsilon}$ then S is subcomplete. The stronger conjecture that this remains true if we only assume $s_n \leqslant \frac{1}{2} n^2$ for large n is still unproved. On the other hand, counterexamples are known [Er (62) a] which satisfy $s_n < \frac{1}{2} n^2 + cn$ for a certain fixed c.

For sequences $S(p) = (p(1), p(2), \ldots)$ with $p(x) \in \mathbf{Z}[x]$, it was shown by Cassels [Cas (60)] that the obvious necessary conditions for the completeness of $S(p)$ are sufficient, namely, $p(x)$ should have positive leading coefficient and $\gcd(p(1), p(2), \ldots) = 1$. This was extended to $p(x) \in \mathbf{R}[x]$ by Graham [Gr (64) b]. Of course, the conditions show in fact that $S(p)$ is strongly complete in this case.

It also follows from Cassels' arguments that if $f(x) = \sum_{k=0}^{n} a_k x^k$ is a monic polynomial defining a P-V number then any sequence $S = (s_1, s_2, \ldots)$

satisfying the linear recurrence $\sum_{k=0}^{n} a_k s_{t+k}$, $t \geqslant 0$, is not strongly complete. This includes, for example, the case $s_n = F_n$, the n^{th} Fibonacci number, defined by $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n, n \geqslant 0$, as well as sequences formed by taking bounded repetitions of the F_n, i.e., $(\overbrace{F_1, \ldots, F_1}^{k}, \overbrace{F_2, \ldots, F_2}^{k}, \ldots)$. However, if each term of this sequence is repeated often enough (depending on the term) then it is strongly complete. In particular, it has been shown by the authors [Er-Gr (72) b] that the sequence

$$S^* = (\overbrace{F_1, \ldots, F_1}^{m_1}, \ldots, \overbrace{F_n, \ldots, F_n}^{m_n}, \ldots)$$

where m_n/ϕ^n decreases and $\phi = \dfrac{1+\sqrt{5}}{2}$, is strongly complete if and only if

$$\sum_{n=1}^{\infty} m_n/\phi^n = \infty .$$

In any case S^* is not strongly complete if

$$\sum_{n=1}^{\infty} m_n/\phi^n < \infty .$$

It is certainly true that similar results must hold for other sequences satisfying linear recurrences although this seems to be a complex subject which is still relatively untouched.

For a complete sequence S, the *threshhold* of *completeness* θ_s is defined to be the least integer θ such that $m \geqslant \theta$ implies $m \in P(S)$. Very little is known about θ_s, even for the relatively simple sequences $S(p) = (p(1), p(2), \ldots)$ with $p(x) \in \mathbf{Q}[x]$. The values of θ_s for $S(p)$ with $p(x) = x^n$ we know [Spr (48) a], [Gr (64) b], [Lin (70)], [Nel (76)] only for $n \leqslant 5$. They are: $\theta_{S(x)} = 1$, $\theta_{S(x^2)} = 128$, $\theta_{S(x^3)} = 12758$, $\theta_{S(x^4)} = 5134240$, $\theta_{S(x^5)} = 67898771$. It seems highly likely that $\theta_{S(x^n)} > \theta_{S(x^{n+1})}$ can occur for infinitely many n. Good candidates should be $n = 2^t$ for large t (or even $t \geqslant 3$?) because of the highly restricted values of m^{2^t} modulo 2^{t+1}.

For a sequence $S = (s_1, s_2, \ldots)$, let $\theta_s(n)$ denote the threshhold of completeness for the truncated sequence $(s_n, s_{n+1}, s_{n+2}, \ldots)$. Even for very simple sequences, the behavior of $\theta_s(n)$ can be very complex. For example, for $S = S(x^2) = (1, 4, 9, \ldots, n^2, \ldots)$, the function $\dfrac{\theta_s(n)}{n^2}$ oscillates in a complicated way between 4 and 5, tending to 5 as $n \to \infty$ for exactly 1487 points between any two consecutive powers of 4 (see [Gr (71)]). In fact, for this case $\theta_s(n)$ is known for almost all values of n (for example, it is always even except for the unique value $\theta_s(3) = 223$ [Gr (∞)]) although the exact determination for all values of n seems very difficult.

For sequences $S = (s_1, s_2, ...)$ with reasonably regular growth, $\theta_s(n)/s_n$ often seems to tend to a limit. For example, for the sequence $P = (p_1, p_2, ...)$ of primes, Kløve [Klø (75)] has conjectured on the basis of computational evidence that $\theta_P(n)/p_n \to 3$ (which would imply the Goldbach conjecture.) It can be shown by probabilistic methods that for any $\alpha \geqslant 2$ there is a sequence $S = (s_1, s_2, ...)$ with $\lim_n \dfrac{\theta_s(n)}{s_n} = \alpha$. It would be interesting to give a construction for each α.

If S is perturbed slightly, its subcompleteness properties are often not affected too severely. For example, if $S^* = (s_1^*, s_2^*, ...)$ with $s_n^* = p(n) + \gamma(n)$ where $p(x) \in \mathbf{Z}[x]$ and $\gamma(n) = O(n^{1-\varepsilon})$ for $\varepsilon > 0$ then Burr [Burr (xx)] has shown that S^* is subcomplete. It may be that this remains true under the weaker hypothesis that $\gamma(n) = o(n)$ or even $\gamma(n) = O(n)$. It can definitely fail if we only require $\gamma(n) = o(n^{1+\varepsilon})$ for any fixed $\varepsilon > 0$. In fact, it can be shown that for any sequence $A = (a_1, a_2, ...)$ and any function $f(n)$ with $\sum_{n \geq 1} 1/f(n) < \infty$, there is a sequence $A' = (a_1', a_2', ...)$ with $|a_n - a_n'| < f(n)$ for n sufficiently large which is not subcomplete. It is not known if there is a sequence $S = (s_1, s_2, ...)$ with $\lim \dfrac{s_{n+1}}{s_n} = 2$ so that $P(S')$ has density one for every cofinite subsequence S' of S. It is known [Burr (71)] that for any $\varepsilon > 0$, if $\dfrac{s_{n+1}}{s_n} \geqslant \dfrac{1+\sqrt{5}}{2} + \varepsilon$ for large n then S cannot be strongly complete. On the other hand, the constant $\dfrac{1+\sqrt{5}}{2}$ is best possible as the following result shows.

Let $F^* = (f_1^*, f_2^*, ...)$ denote the sequence defined by $f_n^* = F_n - (-1)^n$ where F_n is the n^{th} Fibonacci number. Then it has been shown by Graham [Gr (64) g] that F^* enjoys the following unusual combination of properties:

(i) F^* remains complete after any *finite* set of elements is deleted from it i.e., F^* is strongly complete.

(ii) F^* is no longer complete after any *infinite* set is deleted from it.

Is it true that any sequence $S = (s_1, s_2, \ldots)$ with $\frac{s_{n+1}}{s_n} \geqslant 1 + \varepsilon$ satisfying both (i) and (ii) must have $\lim_n \frac{s_{n+1}}{s_n} = \frac{1+\sqrt{5}}{2}$? It is easy to see that if $\frac{s_{n+1}}{s_n} > \frac{1+\sqrt{5}}{2}$ then (ii) is automatically satisfied. It is not hard to construct very irregular sequences (i.e., $\liminf \frac{s_{n+1}}{2} = 1$, $\limsup \frac{s_{n+1}}{s_n} = \infty$) which satisfy (i) and (ii).

The finite version of this problem is very poorly understood at present. Specifically, for a given sequence S, let $C(n)$ and $N(n)$ denote the following statements:

$C(n)$: If any n elements are removed from S to form S' then S' is complete;

$N(n)$: If any n elements are removed from S to form S' then S' is not complete.

Question: For what values of $m < n$ are there sequences S which satisfy both $C(m)$ and $N(n)$?

For example, $T = (1, 2, 4, \ldots, 2^n, \ldots)$ satisfies $C(0)$ and $N(1)$, while $F = (F_1, F_2, F_3, \ldots) = (1, 1, 2, 3, 5, 8, \ldots)$ satisfies $C(1)$ and $N(2)$. It is not known if there is a sequence satisfying $C(2)$ and $N(3)$.

Let $S(t, \alpha) = (s_1, s_2, \ldots)$ with $s_n = [t\alpha^n]$. For what values of t and α is $S(t, \alpha)$ complete? Even in the range $0 < t < 1$, $1 < \alpha < 2$, the behavior is surprisingly complex. For example, it is known [Gr (64) f] that for any k there exists $t_k \in (0, 1)$ so that $\{ \alpha : S(t_k, \alpha) \text{ is complete} \}$ consists of more than k disjoint line segments. It seems likely that $S(t, \alpha)$ is complete for all $1 < \alpha < \frac{1+\sqrt{5}}{2}$ and all $t > 0$. There seems to be little hope of proving this at present though since we do not even know that $\left[\left(\frac{3}{2}\right)^n\right]$ is odd (or even) infinitely often.

It is possible to consider the question of completeness for sequences of rationals. For example, it is known [Gr (63) b] that the sequence $S = (s_1, s_2, \ldots)$ with $s_n = n + \frac{1}{n}$ is strongly complete. Is it true that if

$p(x) \in \mathbf{Q}(x)$ then the sequence $A = (a_1, a_2, \ldots)$ with $a_n = p(n) + 1/n$ is also strongly complete? Which rational functions $r(x) \in \mathbf{Z}(x)$ force $(r(1), r(2), \ldots)$ to be complete?

Suppose α and β are positive reals with α/β irrational and let S denote the sequence $([\alpha], [\beta], [2\alpha], [2\beta], \ldots, [2^n\alpha], [2^n\beta], \ldots)$. Is S complete? What if 2 is replaced by γ where $1 < \gamma < 2$?

Let us call a sequence $S = (s_1, s_2, \ldots)$ of rationals **Q**-*complete* if every sufficiently large rational belongs to $P(S)$. For example, if every sufficiently large prime and every sufficiently large square belongs to S then $S^{-1} = (1/s_1, 1/s_2, \ldots)$ is **Q**-complete [Gr (64) a] (in fact, all positive rationals belong to $P(S^{-1})$). It was shown in [Er-Ste (63)] that if $\sum_{n=1}^{\infty} \frac{1}{s_n} = \infty$ then there exist $b_n \geqslant s_n$ such that the sequence $(1/b_1, 1/b_2, \ldots)$ is **Q**-complete.

Suppose $S = (s_1, s_2, \ldots)$ is an increasing sequence satisfying $s_{n+1}/s_n \geqslant c > 1$ for some c. Is it possible for $P(S^{-1})$ to contain all the rationals in some interval (α, β), $\alpha < \beta$? It has been conjectured by Bleicher and Erdös that the answer is no.

We close this section with a few problems involving sums of subsets which have a somewhat different flavor.

Let $a_1 < \ldots < a_k \leqslant n$ be a sequence of integers and form all sums $\sum_{i=u}^{v} a_i$. Can one have cn^2 distinct numbers in this set for some $c > 0$? This does not happen for the choice $a_i = i$. What happens if we drop the monotonicity restriction but just insist that the a_i be distinct? Perhaps some permutation of $\{1, 2, \ldots, n\}$ has cn^2 such "interval" sums. How many consecutive integers exceeding n can we represent as $\sum_{i=u}^{v} a_i$? Is it true that for any $c > 0$, we can reach cn? For example, by taking $a_i = i$, we can ususally go to $2n$ (we cannot get a power of 2 by such a sum). However, by leaving out some of the integers we may get past the powers of 2.

Suppose $1 \leqslant a_1 < \ldots < a_k \leqslant n$ is a set of integers with the property that all sums of *consecutive* blocks $\sum_{i=u}^{v} a_i$ are distinct. Erdös and Harzheim have asked how large can k be? Must we have $k = o(n)$? What if we remove the monotonicity constraint and/or the distinctness constraint? Also, what is the least m which is not of the form $\sum_{i=u}^{v} a_i$? Can it be much larger than n? We can show that

$$\sum_{x < a_i < x^2} \frac{1}{a_i} < c .$$

Is it true that $\sum_i \frac{1}{a_i}$ is bounded, independent of n?

Is there a sequence $1 \leqslant a_1 < a_2 < \ldots$ so that the number of representations of n as a sum of consecutive a_i's is always positive? Can it tend to infinity with n?

Erdös and Moser [Mo (63)] considered the case where the a_i are the primes. They conjectured that the lim sup of the number of representations is infinite and also that the density of integers which have exactly k representations exists. We do not even know that the upper density of the integers with at least one such representation is positive.

Andrews [An (75)] has recently studied the following related problem of MacMahon. Let $a_1 < a_2 < \ldots$ be an infinite sequence where $a_1 = k$ and a_{i+1} is the least integer which is not a sum of consecutive earlier a_j's. What can be said about the density of the a_i's?

Let $f(n)$ denote the least integer so that one can divide the set of integers $\{1, 2, \ldots, n\}$ into $f(n)$ classes so that n cannot be expressed as a sum of distinct elements from the same class. It can be shown that $f(n) \to \infty$ but we have no idea how fast.

How many integers less than n/k can one give so that n is not a sum of a subset of these integers (where k is fixed and n is large)? Does this depend on n in an irregular way?

For the sequence $0 < a_1 < \ldots < a_n$, let $F_n(t)$ denote the number of solutions of $\sum_{i=1}^{n} \varepsilon_i a_i = t$, $\varepsilon_i = 0$ or 1. Erdös and Moser (see [Kat (66)]) proved that $F_n(t) < \frac{c 2^n}{n^{3/2}} (\log n)^{3/2}$; they conjectured that the factor $(\log n)^{3/2}$ could be omitted and this was proved by Sárközy and Szemerédi [Sár-Sz (65)]. Stanley [Stan (xx)] recently showed that max $F_n(t)$ is assumed if the a_i's form an arithmetic progression and $t = \frac{1}{2} \sum_{i=1}^{n} a_i$ (see also [Lint (67)]).

Finally, suppose $\alpha_1 < \ldots < \alpha_k < x$ is a sequence of *real* numbers with k maximal so that any two sums $\sum_{i=1}^{k} \varepsilon_i \alpha_i$, $\varepsilon_i = 0$ or 1, differ by at least 1. Is it true that $k \leqslant \frac{\log x}{\log 2} + O(1)$? This generalizes the old (and still unsolved) conjecture of Erdös which asks the same question when the α_i are integers. The best result currently available for the integer problem, due to Erdös and Moser, is that

$$k \leqslant \frac{\log x}{\log 2} + \frac{\log \log x}{2 \log 2} + O(1).$$

On the other hand, Conway and Guy [Con-Gu (69)] have shown that for $n \geqslant 21$, it is possible to find $n + 2$ numbers $\leqslant 2^n$ with all distinct subset sums, which is one greater than obtained by just choosing the numbers $1, 2, 4, ..., 2^k, ..., 2^n$ (see [Er+3 (64)], [Gor-Ru (60)], [Er-Sz (76) b] for related results).

It was conjectured by Erdös and proved by C. Ryavec that if $1 \leqslant a_1 < a_2 < ... < a_n$ is a set of integers with all subset sums distinct (i.e., $| P(A) | = 2^n$) then $\sum_{i=1}^{n} \frac{1}{a_i} < 2$. This was recently strengthened by Hanson, Steele and Stenger [Hanson-St-St (77)] who showed $\sum_{i=1}^{n} \left(\frac{1}{a_i}\right)^s < \frac{1}{1 - 2^{-s}}$ for all real $s \geqslant 0$.

7. Irrationality and transcendence

Liouville was the first to prove the existence of transcendental numbers. Looking back from our position of relative "wisdom" it now seems strange that Euler did not discover them. Cantor then proved that the algebraic numbers are denumerable and the reals are not—one of the great new insights which very few humans had the ability and luck to experience first hand (see the famous Cantor-Dedekind correspondence [Can-Ded (37)], [Cav (62)] which should surely be translated into English). The first proof of a well known constant to be transcendental was due to Hermite who proved that e is transcendental. This was followed shortly thereafter by Lindemann's proof of the transcendence of π in 1882. Very much is known now, due to the fundamental work of people such as Siegel, Kusmin, Gelfond, Schneider, Baker, Schanuel, Mahler, Waldschmidt, Sprindzuk and many others (if we omit people this does not mean that we think they are less good than the ones mentioned). However, there are still surprising gaps in our knowledge, e.g., as far as we know $e + \pi$, Euler's constant γ and $\zeta(2n+1)$, $n \geqslant 2$, could all be rational. Very recently, an ingenious argument (see [Poo (79)]) using only continued fraction and binomial coefficient identities has been given by Apéry which shows that $\zeta(3)$ is irrational. We will discuss here only special series which do not connect up with the general theory at all but which seem attractive to us and where often clever special methods are needed which usually are not available in general.

It has been known for some time (see [Er (75)]) that the sum $\sum_{k=1}^{\infty} \frac{1}{2^{n_k}}$ is transcendental if $n_1 < n_2 < \ldots$ is a sequence which increases rapidly enough, e.g., $\limsup_k n_k/k^t = \infty$ for every t is sufficient. As far as we know the weaker condition

$$\limsup_k n_k/k = \infty$$

suffices. On the other hand, we do not know any algebraic number for which

$$\limsup_k (n_{k+1} - n_k) = \infty$$

but one would certainly expect that $\sqrt{2}$ is such a number. In fact, perhaps all algebraic irrationals are normal. There seems to be some hope of proving that if $n_k > ck^2$ then $\sum_{k=1}^{\infty} \frac{1}{2^{n_k}}$ is not the root of any quadratic polynomial.

As usual, let $d(n)$ and $v(n)$ denote the number of divisors of n and the number of prime factors of n, respectively. It is known [Er (48) a] that $\sum_n \frac{d(n)}{2^n}$ is irrational but it is very annoying that at present $\sum_n \frac{v(n)}{2^n}$ cannot be proved irrational. This leads to several interesting conjectures, e.g., are there infinitely many n so that for some c and every i,

$$v(n+i) < ci \ ?$$

We just know too little about sieves to be able to handle such a question ("we" here means not just us but the collective wisdom (?) of our poor struggling human race).

It is not too hard to prove that $\sum_n \frac{1}{2^{\phi(n)}}$ and $\sum_n \frac{1}{2^{\sigma(n)}}$ are irrational but the irrationality of $\sum_n \frac{\phi(n)}{2^n}$ and $\sum_n \frac{\sigma(n)}{2^n}$ is probably hopeless to prove at present (see [Er (57)]).

A few other results of this type (see [Er (58)], [Er (68)], [Op (68)], [Op (71)]: (i) $\sum_n \frac{p_n}{n!}$ is irrational as is $\sum_n \frac{p_n^k}{n!}$ for every k, where p_n denotes the n^{th} prime; on the other hand the irrationality of $\sum_n \frac{p_n}{2^n}$ is probably hopeless; it is known [Er-Pom (78)] that $\sum_n \frac{\varepsilon_n}{2^n}$ is irrational where $\varepsilon_n = 1$ if $P(n+1) > P(n)$ and 0 if $P(n+1) < P(n)$, and $P(n)$ denotes the largest prime factor of n.

(ii) $\sum_n \frac{\sigma_k(n)}{n!}$ is irrational for $k = 1$ and 2 but this is not known for any $k > 2$, where $\sigma_k(n)$ denotes the sum of the k^{th} powers of the divisors of n;

(iii) $\sum_n \frac{1}{2^n - 1} = \sum_n \frac{d(n)}{2^n}$ is known to be irrational; however, neither $\sum_n \frac{1}{2^n - 3}$ nor $\sum_n \frac{1}{n! - 1}$ are known to be irrational. Perhaps $\sum_{k=1}^{\infty} \frac{1}{2^{n_k} - 1}$ is irrational for any $n_1 < n_2 < \ldots$.

Probably $\sum_n \frac{d(n)}{a_1 \ldots a_n}$ is irrational if $a_n \to \infty$ but this has only been proved (by Erdös and Straus [Er-Str (71) b], [Er-Str (74)]) if $a_n \geqslant a_{n-1}$ is assumed. Perhaps there is a constant c so that for infinitely many n,

$$d(n+i) < ci \quad \text{for all} \quad i \geqslant 1 .$$

This inequality would help in the irrationality proof but if true it will be infinitely hard to prove (see [Er (75)], [Str-Sub (xx)]).

The sum $\sum_n \frac{a_n}{2^{a_n}}$ should be irrational if $\frac{a_n}{n} \to \infty$. It can be shown that if $a_{n+1} - a_n \to \infty$ then $\sum_n \frac{a_n}{2^{a_n}}$ is irrational. In fact, we know of no series $\sum_n \frac{a_n}{2^{a_n}}$ being rational for which

$$\limsup_n \, (a_{n+1} - a_n) = \infty$$

but such series probably exist. The sum $\sum_n \frac{a_n}{2^{a_n}}$ is known to be irrational under the stronger hypothesis that

$$a_n > cn \sqrt{\log n \log \log n} .$$

The sum $\sum_q \frac{q}{2^q}$ where q ranges over the squarefree numbers, should be irrational but we do not see a proof at present. In considering this question we were led to the following problems. Does the equation

$$\frac{n}{2^n} = \sum_{k=1}^{t} \frac{a_k}{2^{a_k}}, \quad t > 1,$$

have a solution for infinitely many n? For all n? Is there a rational x for which

$$x = \sum_{k=1}^{\infty} \frac{a_k}{2^{a_k}}$$

has two solutions?

If $a_n^{1/2^n} \to \infty$ then it is not hard to show that $\sum_n \frac{1}{a_n}$ is irrational; the following related concept is of some interest. Let us call a sequence $a_1 < a_2 < \ldots$, an *irrationality sequence* if for all integer sequences $\{t_n\}$ with $t_n \geqslant 1$, the sum

$$\sum_{n=1}^{\infty} \frac{1}{t_n a_n}$$

is irrational. For example, the sequence $a_n = 2^{2^n}$ is an irrationality sequence whereas $a_n = n!$ is not. It would be nice to find a slowly increasing sequence with this property. If a_n is an irrationality sequence then $a_n^{1/n} \to \infty$. If the a_k do not increase too rapidly then for every sufficiently small $\alpha > 0$ there is a suitable choice of the t_n so that

$$\sum_n \frac{1}{t_n a_n} = \alpha .$$

For example, this happens if $a_n < c^n$ for all n.

We might call $a_1, a_2, \ldots$ an irrationality sequence if for every sequence b_n with $b_n/a_n \to 1$, $\sum_n \frac{1}{b_n}$ is irrational. With this definition, we do not even know if $a_n = 2^{2^n}$ is an irrationality sequence. Probably an irrationality sequence of this type must also satisfy $a_n^{1/n} \to \infty$.

Another possibility would be to call a_n an irrationality sequence if for every sequence b_n with $|b_n| < C$, $\sum_n \frac{1}{a_n + b_n}$ is always irrational. In this case, 2^{2^n} is an irrationality sequence although we do not know about 2^n or $n!$ Is there an irrationality sequence of this type with $a_n < n^k$? A related result is the theorem of Erdös [Er (75)]: If $\limsup_n a_n^{1/2^n} = \infty$ and $\sum_n \frac{1}{a_n}$ is rational then for every $\varepsilon > 0$ there are infinitely many m with $a_m < m^{1+\varepsilon}$.

We asked: If $a_n \to \infty$ (fast but not too fast [1])) is it true that not both $\sum_n \frac{1}{a_n}$ and $\sum_n \frac{1}{1+a_n}$ can be rational? D. Cantor observed that this holds for $a_n = \binom{n}{2}$; up to now this is the most rapidly growing sequence which has this rationality property.

If 1 is replaced by a larger constant then higher degree polynomials can be used. For example, if $p(x) = x^3 + 6x^2 + 5x$ then both $\sum_{u \geq 1} \frac{1}{p(n)}$ and $\sum_{n \geq 1} \frac{1}{p(n)+8}$ are rational (since both $p(n)$ and $p(n) + 8$ completely split over the integers). Similar examples are known using polynomials with degrees as large as 10 (see [Har-Wri (60)]). Whether there is a polynomial $f(x)$ of degree exceeding 10 so that for some m, both $f(x)$ and $f(x) + m$ completely split over the integers is not known.

The following pretty conjecture is due to Stolarsky:

$$\sum_{n=1}^{\infty} \frac{1}{a_n + t}$$

cannot be rational for every positive integer t. Unfortunately, we could get nowhere with this conjecture.

It is not too hard to show that if $a_k \to \infty$ rapidly enough then $\sum_k \frac{1}{a_k a_{k+1}}$ is irrational; in fact, $a_k^{1/2^k} > 1 + \varepsilon$ should be enough. It would be interesting to know what the strongest theorem of this type is. Is it true that if $\frac{a_{n+1}}{a_n^2} \to 1$ then $\sum_n \frac{1}{a_n}$ is irrational unless $a_{n+1} = a_n^2 - a_n + 1$ for $n \geqslant n_0$ (see [Er (68)], [Er-Str (63)])?

It has been noted that for the sequence of Fibonacci numbers F_n defined by $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$, $n \geqslant 0$, we have [Goo (74)], [Hog-Bi (76)]

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7 - \sqrt{5}}{2}$$

However, nothing is known about the character of the related sums

$$\sum_{n=1}^{\infty} \frac{1}{F_{2^n+1}} \text{ or } \sum_{n=0}^{\infty} \frac{1}{L_{2^n}}$$

---

[1]) Half-fast ?

where $L_n = F_{n-1} + F_{n+1}$. Is it true that $\sum_{k=1}^{\infty} \frac{1}{F_{n_k}}$ is irrational for any sequence $n_1 < n_2 < \dots$ with $\frac{n_{k+1}}{n_k} \geqslant c > 1$? Is it enough to have $\frac{n_k}{k} \to \infty$? Also, what about $\sum_{n=1}^{\infty} \frac{1}{F_n}$?

Let Q be a set of primes (possibly infinite) and let $a_1 < a_2 < a_3 < \dots$ denote the set of integers all of whose prime factors belong to Q. The sum

$$\sum_{n=1}^{\infty} \frac{1}{\text{lcm}\ (a_1, \dots, a_n)}$$

turns out to be irrational if Q is infinite. What happens for finite Q (with more than one element)?

Erdös and Straus [Er-Str (∞)] proved that if one takes all sequences of integers $a_1, a_2, \dots$, with $\sum_k \frac{1}{a_k} < \infty$, then the set

$$\left\{(x, y) : x = \sum_k \frac{1}{a_k}, y = \sum_k \frac{1}{1 + a_k}\right\}$$

contains an open set. Is the same true in three (or more) dimensions, e.g., taking all (x, y, z) with

$$x = \sum_k \frac{1}{a_k}, \quad y = \sum_k \frac{1}{1+a_k}, \quad z = \sum_k \frac{1}{2 + a_k}\ ?$$

Irrationality often could be deduced if we knew more about diophantine equations. For example, here is a typical (though somewhat artificial) question. Set

$$A_n = \text{lcm}\ (1, \dots, n)$$

and put $A_n' = \left(\prod_{\substack{\text{prime} \\ p \leq n}} p \right) A_n$. Is $\sum_n \frac{1}{A_n'}$ irrational? This would follow immediately if we could show that

$$q^2 - p^3 < q^{1-\varepsilon}$$

has infinitely many solutions in primes p and q.

It seems that series like $\sum_{n=1}^{\infty} \left(\prod_{i=1}^{n} (n + i) \right)^{-1}$ are very hard to treat, though they surely are irrational. However, it is known [Hansen (75)], for example, that

$$\sum_{n \geq 1} \frac{1}{\binom{2n}{n}} = \frac{1}{3} + \frac{2\pi}{3^{5/2}}$$

$$\sum_{n \geq 1} \frac{(-1)^{n-1}}{\binom{2n}{n}} = \frac{1}{5} + \frac{4}{5^{3/2}} \log \left(\frac{1+\sqrt{5}}{2} \right),$$

and so, are transcendental. Let $f(n) \to \infty$ as $n \to \infty$. Is it true that

$$\sum_{n=1}^{\infty} \left(\prod_{1 \leq i \leq f(n)} (n+i) \right)^{-1}$$

is irrational? The answer is almost surely in the affirmative if $f(n)$ is assumed to be nondecreasing but at present we lack methods to decide such questions.

8. Diophantine problems

In this section we will discuss a variety of questions which can be loosely classified under the category of "diophantine" problems. We will not dwell on the classical problems here; there are many excellent books available on the subject. Rather, we will mainly discuss special or unconventional diophantine problems which none the less appear to be not without interest.

First, let us mention that several old problems have recently finally been settled. The first of these is the conjecture of Catalan, which asserts that 8 and 9 are the only consecutive powers. Tijdeman [Ti (76)] proved that there do not exist two consecutive powers exceeding a large but computable number and it seems likely that Catalan's conjecture will be completely proved in the near future. If $A = \{ a_1 < a_2 < \ldots \}$ denotes the set of powers then Choodnowski further claims to have proved that there is a computable function $f(n)$ tending to infinity with n so that $a_{n+1} - a_n > f(n)$. There seems little doubt that $f(n) > c_1 n^{c_2}$ but this seems hopeless at present.

The second old problem which finally succumbed (to repeated attacks by Erdös and Selfridge [Er-Se (75)], [Er (55)]) was the conjecture that the product of consecutive integers is never a power. It is not difficult to prove that for any b, there are only finitely many sequences $0 < a_1 < a_2 < \ldots < a_t$ with $a_{i+1} - a_i < b$ so that $a_1 a_2 \ldots a_t$ is a power. It is known that the related binomial coefficient $\binom{n}{k}$ is never a power for $k \geqslant 4$ and $n \geqslant 2k$. Of course $\binom{n}{2}$ is a square infinitely often; Tijdeman's methods will probably give $\binom{n}{2} \neq x^l$, $l > 2$ and $\binom{n}{3} = x^l$ implies $n = 50$, $l = 2$ (this is known to be the only solution for $l = 2$).

In the same spirit one could ask when the product of two or more disjoint blocks of consecutive integers can be a power. For example, if $A_1, \ldots, A_n$ are disjoint intervals each consisting of at least 4 integers then perhaps the product $\prod_{k=1} \prod_{a_k \in A_k} a_k$ is a nonzero square in only a finite number of cases. Pomerance has pointed out that the product of the four blocks of 3 consecutive integers $(2^{n-1}-1)\, 2^{n-1}\, (2^{n-1}+1)$, $(2^n-1)\, 2^n\, (2^n+1)$, $(2^{2n-1}-2)\,(2^{2n-1}-1)\, 2^{2n-1}$ and $(2^{2n}-2)\,(2^{2n}-1)\, 2^{2n}$ is the square of $2^{3n}\,(2^{2n-2}-1)\,(2^{2n-1}-1)\,(2^{2n}-1)$.

We now mention a few problems on the prime factors of consecutive integers (also see [Er (75) a*] and [Er-Str (77)]). Set

$$A(n,k) = \prod_{\substack{p^\alpha \| n \\ p \leq k}} p^\alpha$$

where, as usual, p denotes a prime. Mahler [see [Rid (57)]) proved that for every fixed k and l,

$$\prod_{i=1}^{l} A(n+i,k) < n^{1+\varepsilon} \tag{1}$$

for each $\varepsilon > 0$ provided $n > n_0(\varepsilon, k, l)$. On the other hand it is easy to see that for a certain $c > 0$

$$A(n,3)\, A(n+1,3) > cn \log n \tag{2}$$

for infinitely many n. The estimate in (1) should be improved but this will almost certainly be difficult. On the other hand, it should not be too hard to improve (2), e.g.,

$$\limsup_n A(n,3)\, A(n+1,3)/n \log n \to \infty .$$

However, the determination of the exact maximal order of $A(n, 3)\, A(n+1, 3)$ will no doubt be very difficult.

Let us set

$$f(n,k) = \min_{1 \leq j \leq k} A(n,j) .$$

It is easy to see by an averaging process that

$$f(n,k) < ck.$$

An old conjecture of Erdös asserts that for every $\varepsilon > 0$ there is a $k_0(\varepsilon)$ so that for $k > k_0(\varepsilon)$

$$\max_{1 \leq n < \infty} f(n,k)/k \to 0 \text{ as } k \to \infty.$$

It would be of great interest to determine the exact order (in k) of $\max_{1 \leq n < \infty} f(n,k)$—we do not even have a plausible conjecture. It is not hard to see that it tends to infinity with k.

Let $B_k(n)$ denote the quantity

$$B_k(n) = \prod_{\substack{p^\alpha \| n \\ \alpha \geq k}} p^\alpha .$$

Erdös asked Mahler more than 2.5×10^9 years ago whether there are infinitely many integers n, $n+1$ with

(3) $$B_2(n) = n, \; B_2(n+1) = n+1.$$

Mahler immediately observed that the answer is yes, since $x^2 - 8y^2 = 1$ has infinitely many solutions. However, the system

$$B_2(n) = n, \; B_2(n+1) = n+1, \; B_2(n+2) = n+2.$$

almost certainly has no solution. Is it true that all (or all but a finite number of) the solutions of (3) come from Pellian equations and that the number of $n < x$ satisfying (3) is at most $(\log x)^c$?

Is it true that

$$B_2(n) = n, \; B_3(n+1) = n+1 ?$$

has no solution? Is $n = 8$ the only solution for

$$B_3(n) = n, \; B_2(n+1) = n+1 ?$$

(Also, see [Gol (70)] for related results).

Set

$$G(n,k) = \prod_{i=1}^{k} B_2(n+i).$$

Perhaps

$$G(n,k) < c_k n^2$$

but we cannot even disprove that infinitely often

$$G(n, k) = c_k \prod_{i=1}^{k} (n+i)$$

for every k. It seems very likely that

$$G(n, k) < n^{2+\varepsilon}$$

for every k and $\varepsilon > 0$. It would be interesting to obtain nontrivial upper and lower bounds for $\prod_{i=1}^{k} B_r(n+i)$ for $r > 2$. Is it true that

$$\lim_n \sup \prod_{i=1}^{k} B_r(n+i)/n^{1+\varepsilon} \to \infty$$

for every r and $k > 1$? This is not even clear for $r = 3$.

Denote by $P(n)$ the largest prime factor of n. Classical results state that if $f(n)$ is a polynomial which is not a power of a linear polynomial then $P(f(n)) \to \infty$ as $n \to \infty$. In particular, it is known [Mah (35)] (also see [Kot (73)]) that

$$P\big(n(n+1)\big) > c \log \log n\,.$$

No doubt, this can be substantially improved. There are heuristic reasons for believing that the right order of magnitude is $(\log n)^2$. Schinzel [Sch (67)a] observed that for infinitely many n

$$P\big(n(n+1)\big) < n^{c/\log \log \log n}\,.$$

Is it true that for every $n \geqslant n_0(\varepsilon)$ there are two (or more generally k) consecutive integers less than n, all of whose prime factors are less than n^{ε}? The answer should be affirmative but the problem seems very hard. Similarly, one can ask if there are infinitely many n so that

$$P(n) < \sqrt{n}\,,\; P(n+1) < \sqrt{n+1}$$

or more generally,

$$P(n+i) < n^{1-\varepsilon},\; 1 \leqslant i \leqslant k\,.$$

We know very little about this. Pomerance has pointed out that

$$P(n) > n^{e^{-1/2}-\varepsilon},\, P(n+1) > (n+1)^{e^{-1/2}-\varepsilon}$$

has solutions by density considerations.

In a similar vein, is it true that every $n > n_0(\varepsilon)$ can be written in the form $a + b$ with

$$P(a) < n^{\varepsilon}, \quad P(b) < n^{\varepsilon} \ ?$$

It is surprising how unexpectedly difficult the question is.

Erdös and Pomerance [Er-Pom (78)] showed that if $f(\delta)$ denotes the upper density of the set of n with

$$n^{-\delta} < P(n)/P(n+1) < n^{\delta}$$

then $f(\delta) \to 0$ as $\delta \to 0$. Further, they show that $P(n) < P(n+1) > P(n+2)$, $P(n) > P(n+1) < P(n+2)$ and $P(n) < P(n+1) < P(n+2)$ each have infinitely many solutions but they cannot prove the same for $P(n) > P(n+1) > P(n+2)$. They conjecture that the set of n with $P(n) > P(n+1)$ has density 1/2 but can only prove that it and its complement each have positive upper density.

If it were known that $P(n(n+1))/\log n \to \infty$ as $n \to \infty$ then it would follow that the equation

$$n! = \prod_i a_i!\,, \quad n > a_1 \geqslant a_2 \geqslant \dots \tag{4}$$

could have only finitely many nontrivial solutions (see [Er (75) a*]). By nontrivial here we mean that $a_1 \leqslant n - 2$ since otherwise we can set $n = a_2!a_3!\dots$ Hickerson conjectures that the largest nontrivial solution to (4) is

$$16! = 14!\,5!\,2!$$

The equation

$$a_1!\,a_2!\dots a_t! = y^2 \tag{5}$$

has been studied recently in [Er-Gr (76)]. Define F_k by

$$F_k = \{\, m : \text{for some } A \subseteq \{1, 2, \dots, m\} \text{ with } m \in A \text{ and } |A| \leqslant k,\ \prod_{a \in A} a! = y^2 \text{ for some integer } y \,\}$$

and set

$$D_k = F_k - F_{k-1}$$

Of course, if p is prime then $p \notin D_k$ for any k. If for a set $S \subseteq \mathbf{Z}^+$ we let $S(n)$ denote $|S \cap \{1, 2, \dots, n\}|$, then it is known that:

(i) $D_2 = \{ n^2 : n > 1 \}$;

(ii) $D_3(n) = o(D_4(n))$;

(iii) If $p \in \{2, 3, 5, 7, 11\}$ is a proper divisor of n then $n \in F_5$;

(iv) For almost all primes p, $13p \notin F_5$;

(v) The least element in D_6 is $527 = 17 \cdot 31$;

(vi) $D_k = \varnothing$ for $k > 6$.

We still do not know the order of growth of $D_4(n)$, for example. It seems likely that $D_6(n) > cn$ but this isn't known. For other results and questions of this type, the reader can consult [Er (75) a*], [Ec+3 (xx)], [Ec-Eg (72)].

It was conjectured by Grimm [Gri (69)] that if $n+1, \ldots, n+k$ are consecutive composite numbers then there is a set of k distinct primes p_i so that $p_i \mid n+i$, $1 \leqslant i \leqslant k$.

This conjecture is certainly very deep since as observed by Erdös and Selfridge, it would imply that there is always a prime between any two consecutive squares. The strongest results in this direction are the results of Ramachandra, Shorey and Tijdeman [Ram-Sh-Ti (75)]. They show that Grimm's conjecture holds for $n > n_0$ provided $k < c(\log n/\log\log n)^3$.

Is it true that $\binom{2n}{n}$ is never squarefree if $n > 4$? It is annoying that this problem is difficult. More generally, denote by $f(n)$ the largest integer so that for some prime p, $p^{f(n)}$ divides $\binom{2n}{n}$. The quantity $f(n)$ should tend to ∞ as $n \to \infty$ but this is not known. It is known that $n = 23$ is the largest value of n for which all $\binom{n}{k}$ are squarefree for $0 \leqslant k \leqslant n$. On the other hand, we cannot even disprove $f(n) > c \log n$.

It is known [Er+3 (75)] that for any two primes p and q, there are infinitely many n for which $\left(\binom{2n}{n}, pq\right) = 1$. However, for three primes we know almost nothing. For example, is $\left(\binom{2n}{2}, 105\right) = 1$ infinitely often? Computation of V. Vyssotsky has shown that the least odd prime factor of $\binom{2n}{n}$ is 13 for $n = 3160$ and at most 11 for $3160 < n < 10^{90}$. This has been extended to 5.3×10^{100} by Kimble [Ki (79)]. If we set

$$f(n) = \sum_{\substack{p \nmid \binom{2n}{n} \\ p \leq n}} \frac{1}{p}$$

then we cannot decide if $f(n)$ is unbounded. We have shown [Er+3 (75)] that

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n=1}^{x} f(n) = \sum_{k=2}^{\infty} \frac{\log k}{2^k} = \gamma_0$$

and

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n=1}^{x} f^2(n) = \gamma_0^2$$

so that for all but $o(n)$ integers $m \leqslant n$, $f(m) = \gamma_0 + o(1)$.

It seems likely that the density of integers n for which $\binom{n}{k}$ is squarefree for at least r values of k, $1 \leqslant k \leqslant n-1$, has a density $c_r > 0$ and that $\sum_{r=0}^{\infty} c_r = 1$. We can prove that for k fixed and large, the density of n such that $\binom{n}{k}$ is squarefree is less than ε_k where $\varepsilon_k \to 0$ as $k \to \infty$. Also, there exist infinitely many n such that $\binom{n}{k}$, $1 \leqslant k \leqslant n-1$, is never squarefree. Probably their density is positive.

For given n, let $s(n)$ denote the largest integer such that for some k, $\binom{n}{k}$ is divisible by the $s(n)^{\text{th}}$ power of a prime. It is easy to see that $s(n) \to \infty$ as $n \to \infty$. In fact it is not hard to show that $s(n) > c \log n$, which is the right order of magnitude. However, if $S(n)$ denotes the largest integer so that for *all* k, $1 \leqslant k < n$, $\binom{n}{k}$ is divisible by the $S(n)^{\text{th}}$ power of some prime, then it is quite likely that $\limsup S(n) = \infty$ although this is not known at present.

It is probably true that

$$P\left(\binom{n}{k}\right) > \max\{n-k,\, k^{1+\varepsilon}\}$$

but this seems very deep (cf. [Ram (70)], [Ram (71)]). It is not hard to prove

$$P\left(\binom{n}{k}\right) > ck \log k\,.$$

For $k^{1+\varepsilon} < n$ it should be true that

$$P\left(\binom{n}{k}\right) > k^{1+\varepsilon} .$$

Let us abbreviate $\prod_{i=o}^{v-u} (u+i)$ by $\pi(u, v)$. An integer n is called *bad* if it belongs to some interval $u \leqslant n \leqslant v$ so that the greatest prime factor $P(\pi(u, v))$ occurs in $\pi(u, v)$ with an exponent greater than 1 (see [Er-Gr (76)] for applications of this concept). Let $B(x)$ denote the number of bad integers not exceeding x. Is it true that $B(x)$ is asymptotic to $B'(x)$, the number of integers $n \leqslant x$ for which $P(n)^2 \mid n$? It can be shown that

$$B'(x) = \frac{x}{\exp\left((c+o(1))\ (\log x \log\log x)^{1/2}\right)}$$

We only know that $B(x) > x^{1-\varepsilon}$ for any $\varepsilon > 0$. An integer n is called *very bad* if for some u and v, with $u \leqslant n \leqslant v$, $\pi(u, v)$ is powerful, i.e., $p \mid \pi(u, v)$ implies $p^2 \mid \pi(u, v)$. The number of very bad numbers less than x should be less than $cx^{1/2}$ and, in fact, probably is asymptotic to the number of powerful numbers less than x but this is not known.

As we noted earlier, $n(n+1)$ is powerful for infinitely many n. Erdös and Selfridge have conjectured that the product of more than two consecutive numbers is never powerful.

If $P(\pi(u, v))^2 \mid \pi(u, v)$ then $v - u$ must be relatively small. It follows from results of Ramachandra that it is at most $v^{1/2-\varepsilon}$ but no doubt it is in fact bounded by v^{ε} for every $\varepsilon > 0$. Certainly it can be arbitrarily large although this has not been proved. Is it true that for every k there are infinitely many values of p such that

$$P\left(\prod_{i=0}^{k} (p^2+i)\right) = p\ ?$$

If $p(x)$ denotes the least prime factor of x then Ecklund [Ec (69)] proved that

$$p\left(\binom{n}{k}\right) < \frac{n}{2} \text{ for } k > 1$$

with the unique exception of $p\left(\binom{7}{3}\right) = 5$, thus settling a conjecture of Erdös and Selfridge. They also conjectured that

$$p\left(\binom{n}{k}\right) < \frac{n}{k} \text{ for } n > k^2 ;$$

they have proved that

$$P\left(\binom{n}{k}\right) < \frac{c_1 n}{k^{c_2}} .$$

Define $F(n)$ by

$$F(n) = \max_{\substack{m<n \\ m \neq \text{prime}}} (m + p(m)) .$$

Is $F(n) > n$ for $n > n_0$? Does $F(n) - n \to \infty$ as $n \to \infty$?

Can $\binom{n}{k}$ be the product of consecutive primes infinitely often? For example,

$$\binom{21}{2} = 2 \cdot 3 \cdot 5 \cdot 7 .$$

A proof that this cannot happen infinitely often for $\binom{n}{2}$ seems hopeless; probably this can never happen for $\binom{n}{k}$ if $3 \leqslant k \leqslant n - 3$.

Erdös once conjectured that $\binom{n}{k}$ always has a divisor of the form $n - i$ for some $i < k$. This was disproved by Schinzel and Erdös (see [Sch (58)] where further problems of this type are stated). Is it true that there is an absolute constant c so that $\binom{n}{k}$ always has a divisor in (cn, n)?

Can one classify all solutions of

$$\prod_{i=1}^{k_1} (m_1 + i) = \prod_{i=1}^{k_2} (m_2 + i)$$

where $1 < k_1 < k_2$ and $m_1 + k_1 \leqslant m_2$? Perhaps there are only finitely many solutions. More generally, if $k_1 > 2$ then for fixed a and b

$$a \prod_{i=1}^{k_1} (m_1 + i) = b \prod_{i=1}^{k_2} (m_2 + i)$$

should have only a finite number of solutions. What if one just requires that $\prod_{i=1}^{k_1} (m_1 + i)$ and $\prod_{i=1}^{k_2} (m_2 + i)$ have the same prime factors (say, with $k_1 = k_2$)?

Erdös and Straus recently raised the following question: Is it true that for every n there is a k so that

$$\prod_{i=1}^{k} (n+i) \,\Big|\, \prod_{i=1}^{k} (n+k+i) \ ?$$

For example, for $n = 2$ one can choose $k = 4$:

$$3 \cdot 4 \cdot 5 \cdot 6 \,|\, 7 \cdot 8 \cdot 9 \cdot 10 .$$

No example is known for $n \geqslant 10$.

Write

$$(6) \qquad n! = (n+i_1)(n+i_2) \ldots (n+i_k), \quad 0 < i_1 < \ldots < i_k ,$$

where k is variable. Erdös and Selfridge [Er-Se (xx)] have shown that

$$\min i_k = n + cn/\log n$$

which is in fact the right order of magnitude. However, the exact value of c is not known. We know that the largest value of k in (6) is

$$n - \frac{2n \log 2}{\log n} + o\left(\frac{n}{\log n}\right).$$

Define $t(n)$ by

$$t(n) = \max\left\{ a_1 : n! = \prod_{i=1}^{n} a_j , \ a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n \right\}.$$

It has recently be shown by Erdös, Selfridge and Straus (unpublished) that

$$\lim_{n \to \infty} \frac{t(n)}{n} = \frac{1}{e} .$$

Can $t(n) = \dfrac{n}{e} - \dfrac{cn}{\log n}$? Similar results for the case that the a_i are prime powers have been obtained by Alladi and Grinstead [Al-Gri (77)].

It can be shown that the least value $A(n)$ of t for which $n!$ can be written as

$$n! = a_1 a_2 \ldots a_t$$

where $a_k \leqslant n$, $1 \leqslant k \leqslant t$, satisfies

$$A(n) = n - \frac{n}{\log n} + o\left(\frac{n}{\log n}\right).$$

(A proof can be based on a "greedy" decomposition of $n!$, i.e., try to use n as often as possible, then try $n - 1$ as often as possible, etc.). It is not known how long this remains valid if we relax the constraints on the a_k. For example, suppose we only require $a_k \leqslant n\, f(n)$. In particular, what is the situation when $f(n) = n$? In this case is it true that

$$A(n) = \frac{n}{2} - \frac{n}{2 \log n} + o\left(\frac{n}{\log n}\right) ?$$

Another question: Write

$$n! = a_1 a_2 \dots a_t\,, \quad a_1 < \dots < a_t$$

to minimize

$$a_t - a_1$$

(for fixed t or variable t). We do not even know that this cannot be *one* infinitely often.

Let $t_k(n)$ denote the least integer m for which

$$\prod_{i=1}^{k} (m+i) \equiv 0 \pmod{n}.$$

Erdös conjectured and R. R. Hall proved that

$$\frac{1}{x} \sum_{n=1}^{x} t_2(n) \to 0$$

as $x \to \infty$. Probably the factor $\frac{1}{x}$ can be replaced by $\frac{(\log x)^\alpha}{x}$ but this has not yet been proved.

An old conjecture of Erdös asserts that if $x + n \leqslant y$ then

$$\operatorname{lcm}(x+1, \dots, x+n) \neq \operatorname{lcm}(y+1, \dots, y+n)\,.$$

It follows from the Thue-Siegel theorem that for n fixed, $\operatorname{lcm}(x+1, \dots, x+n) = \operatorname{lcm}(y+1, \dots, y+n)$ has only finitely many solutions in x and y. Can one show that for every k there is an n so that

$$\prod_{i=0}^{k} (n-i) \;\Big|\; \binom{2n}{n} ?$$

Of course, $(n+1)$ always divides $\binom{2n}{n}$ but occurrences of n dividing $\binom{2n}{n}$

are quite rare. Are there only finitely many solutions to

$$\prod_i \binom{2m_i}{m_i} = \prod_j \binom{2n_j}{n_j}$$

where the m_i and n_j are distinct?

An old conjecture states that the only solutions of

$$n! = x^2 - 1$$

are $n = 4, 5, 7$. This is almost certainly true but it is intractable at present. Erdös and Obláth [Er-Ob (37)] proved that

$$n! = x^k \pm y^k,\ (x, y) = 1, k > 2$$

has no solutions if $k \neq 4$; Pollack and Shapiro [Pol-Sh (73)] showed that it also has no solutions if $k = 4$. It would be interesting to be able to drop the condition $(x, y) = 1$ but unfortunately, the known methods break down. It is annoying that we cannot even show that for all k there is an n_k so that in the prime decomposition of $n_k!$,

$$n_k! = 2^{\alpha_1} 3^{\alpha_2} \dots p_r^{\alpha_r},$$

all the α_i, $1 \leqslant i \leqslant k$, are even.

It is easy to show that if $\dfrac{n!}{a_1!\,a_2!}$ is an integer then we must have

$$a_1 + a_2 < n + c \log n$$

(although the best value of c is not known). For a fixed k, define $g_k(n)$ to be the largest integer so that for some a_i with

$$n + g_k(n) = a_1 + \dots + a_k,$$

$$\frac{n!}{a_1! \dots a_k!} \text{ is an integer}.$$

Again, it is easy to show that $g_k(n) < c_k \log n$ although the best value of c_k is unknown. Can one show that

$$\sum_{n=1}^{x} g_k(n)/x \log x \to c_k\ ?$$

In fact, it is no doubt true that

$$g_k(n) = c_k \log x + o(\log x)$$

for almost all $n < x$ but we have not proved this.

If we disregard the small primes then the situation probably changes. For example, is it true that we can find $a_1 + a_2 > n + d_r \log n$ with $d_r \to \infty$ as $r \to \infty$ so that

$$\frac{n!\, 2^n \cdot 3^n \ldots p_r^n}{a_1!\, a_2!}$$

is an integer?

The following conjecture (which arose in connection with certain generalizations of van der Waerden's theorem on arithmetic progressions) has attracted some attention during the past 10 years.

Conjecture (Graham [Gr (70)]). For any set A of n positive integers,

$$\max_{a, a' \in A} \frac{a}{(a, a')} \geqslant n\,. \tag{7}$$

We can assume without loss of generality that g.c.d. $\{ a : a \in A \} = 1$. It was then furthermore conjectured that the only sets satisfying (7) with equality are:

(i) $\{ 1, 2, ..., n \}$;

(ii) $\{ L/1, L/2, ..., L/n \}$ where $L = \operatorname{lcm} \{ 1, 2, ..., n \}$;

(iii) $\{ 2, 3, 4, 6 \}$.

It is known at present that (7) holds when:

(a) All $a \in A$ are squarefree. In this case (7) is equivalent to the set-theoretic result of Marica and Schönheim [Mar-Sch (69)] that for any family of n distinct sets $A_1, ..., A_n$, there are at least n distinct differences $A_i - A_j$. This was subsequently generalized in several ways (see [Mar (71)], [Er-Sc (69)]). In particular, Daykin and Lovász [Day-Lo (76)] proved that the number of values taken by any nontrivial Boolean function is not less than the number of sets over which it is evaluated.

(b) $\min_{a \in A} a$ is prime [Win (70)];

(c) n is prime [Sz ($\pm$)];

(d) $n - 1$ is prime [Vé (77)];

(e) For some prime $p > n$, $p \mid a$ for some $a \in A$ [Vé (77)];

(f) For some prime $p > \frac{n-1}{2}$, $p \mid a$ for some $a \in A$ [Boy (78)];

(g) $n = p^2$ for a prime p [Boy (78)];

(h) Some $a \in A$ is prime where $a \neq \frac{1}{2}(a' + a'')$, a', a'' distinct elements of A [Weins (77)];

(i) Some $a \in A$ is prime [Che (xx)], [Pom (78)];

(j) $\min_{a \in A} a$ is p, p^2 or p^3 [Pom (78)];

(k) $n \leqslant 92$ [Che (xx)].

One approach to proving (7) would be to show that the *set* of values $\left\{\frac{a}{(a, a')} : a, a' \in A\right\}$ has at least n elements. Erdös and Szemerédi noted that this is not true by constructing examples for which the set of values has less than n^{1-c} elements. However, they showed it must have at least $n^{c'}$ elements for some $c' > 0$.

P. Frankl asked if the equation

$$\sum_{i=0}^{r} (n+i)! = k^2$$

has only finitely many solutions. He showed that by prime number theory it follows that $r > cn/\log n$. Burr and Erdös then asked whether

$$\sum_i a_i! = 2^m, \ a_1 < a_2 < \ldots$$

has only finitely many solutions. The largest one seemed to be

$$2^7 = 2! + 3! + 5! \ .$$

This was proved to be the largest solution by Frankl [Frank (76)] and independently, by S. Lin. In fact, Lin [Lin (76)] showed somewhat unexpectedly that the largest power of 2 which can divide a sum of distinct factorials containing 2 is 2^{254}. More generally, if

$$p^\alpha \,\|\, (a_1! + a_2! + \ldots + a_k!), \ a_1 < \ldots < a_k,$$

is there a bound $f(a_1, p)$ for α (where, as usual, $p^\alpha \,\|\, n$ means that p^α is the largest power of p dividing n). Conceivably, the answer could depend on a_1 and p. Is there a p and an infinite sequence $a_1 < a_2 < \ldots$ so that

$$p^{\alpha k} \,\|\, \sum_{i=1}^{k} a_i!$$

and $\alpha_k \to \infty$ as $k \to \infty$? Lin also showed that the only solutions to

$$\sum_i a_i! = 3^k$$

are:

$$1! = 3^0, \quad 1!+2! = 3, \quad 1!+2!+3! = 3^2, \quad 1!+2!+4! = 3^3,$$
$$1!+2!+3!+6! = 3^6.$$

Is it true that the equation

$$(p-1)! + a^{p-1} = p^k$$

has only a finite number of solutions (where p is prime). No doubt $(p-1)! + a^{p-1}$, $a > 1$, is rarely a power although $6! + 2^6 = 28^2$ and there may be other solutions.

Is it true that

$$2^n = \sum_i \varepsilon_i 3^i, \quad \varepsilon_i = 0 \text{ or } 1,$$

has only finitely many solutions? $4 = 3 + 1$ and $256 = 3^5 + 3^2 + 3 + 1$ seem to be the only ones. For the analogous question in which only $\varepsilon_i = 1$ or 2 is allowed, is 15 the largest value of n in this case?

Finally, we mention a conjecture of D. J. Newman which illustrates our general ignorance in these matters. If $w(n)$ denotes the number of solutions of

$$n = 2^a + 3^b + 2^c \cdot 3^d,$$

is it true that $w(n)$ is *bounded*?

9. Miscellaneous problems

In this section we will discuss a number of problems which for the most part are even more unconventional than those mentioned up to now. To begin with, we consider some unconventional iteration problems. With $\phi(n) = \phi_1(n)$ denoting the ordinary Euler ϕ-function, define $\phi_k(n)$ to be $\phi(\phi_{k-1}(n))$ for $k \geqslant 2$. The function $f(n)$ defined by

$$f(n) = \min\{k : \phi_k(n) = 1\}$$

was first investigated by Pillai [Pil (33)]. He proved

$$\frac{\log n}{\log 3} < f(n) < \frac{\log n}{\log 2}$$

for n large. H. N. Shapiro [Shap (43)], [Shap (50)] proved that $f(n)$ is essentially multiplicative. An old (and still unresolved) problem of Erdös asks whether or not $f(n)/\log n$ has a distribution function. Is it possible that $f(n)/\log n$ is almost always constant? What can be said about the largest prime factor $P(\phi_k(n))$ of $\phi_k(n)$, e.g., where $k = \log\log n$? Presumably we can have $k \to \infty$ as slowly as we please and have for any $\varepsilon > 0$ and almost all n, $P(\phi_k(n)) = o(n^{\varepsilon})$.

A curious problem of Finucane asks: How many iterations of $n \to \phi(n) + 1$ are needed before a prime is reached? Can it happen that infinitely many n reach the same prime p? What is the density of n which reach p?

One can modify the problem and consider the transformation $n \to \sigma(n) - 1$. Is it true that iterates of this always eventually reach a prime? If so, how soon? Of course, nothing can be proved here but one does seem to reach a prime surprisingly soon. Weintraub [Weint (78)] has found that for $n \leqslant 10^6$, a prime is always reached in fewer than 50 iterations.

If $\sigma_1(n) = \sigma(n)$ and $\sigma_k(n) = \sigma(\sigma_{k-1}(n))$ for $k \geqslant 2$, is $\lim_k \sigma_k(n)^{1/k} = \infty$?

Let $g(n) = n + \phi(n) = g_1(n)$ and $g_k(n) = g(g_{k-1}(n))$ for $k \geqslant 2$. For which n is $g_{k+r}(n) = 2\, g_k(n)$ for all (large) k? The known solutions to $g_{k+2}(n) = 2\, g_k(n)$ are $n = 10$ and $n = 94$. Selfridge and Weintraub both found solutions to $g_{k+9}(n) = 9 g_k(n)$ (all n found were even) and Weintraub also discovered

$$g_{k+25}(3114) = 729 g_k(3114)\,, \quad k \geqslant 6\,.$$

We know of no general rules for forming such examples.

In the 1950's van Wijngaarden raised the following problem: Set $\sigma_1(n) = \sigma(n)$, $\sigma_k(n) = \sigma(\sigma_{k-1}(n))$, $k \geqslant 2$. Is it true that there is essentially only one sequence $\{\sigma_k(n)\}$, i.e., for every m and n, $\sigma_i(m) = \sigma_j(n)$ for some i and j. Selfridge informs us that numerical evidence seems to suggest that this is incorrect. It seems unlikely that anything can be proved about this in the near future.

In view of this situation, Erdös considered iterates of the function $f(n) = n + v(n)$ where $v(n)$ denotes the number of prime factors of n. Here it is overwhelmingly probable that there is essentially only one sequence $\{f_k(n)\}$. This would follow immediately if one could prove that $v(n)$ has infinitely many "barriers", i.e., integers n so that for all $m < n$, $m + v(m) \leqslant n$. This could be attacked by sieve methods but at present these methods are not strong enough. In fact, it does not even seem to be possible to prove the much weaker assertion that for some $\varepsilon > 0$, there are infinitely many n such that $m + \varepsilon v(m) \leqslant n$ for all $m < n$. For earlier work on similar questions, see [Stol (76) b] (especially the references).

Very recently, C. Spiro [Spi (77)] independently raised the following related question. If we iterate the function $h(n) = n + d(n)$, i.e., $h_1(n) = h(n)$, $h_k(n) = h(h_{k-1}(n))$, $k \geqslant 2$ (where as usual $d(n)$ denotes the number of divisors of n) then is there essentially only one sequence $\{ h_k(n) \}$? The answer seems certain to be yes, although here the existence of barriers is much more doubtful. Erdös and Selfridge convinced themselves that if there is a barrier exceeding 24 then it must be extremely large. Spiro also conjectured that if $g(n) = n - d(n) = g_1(n)$, $g_k(n) = g_{k-1}(n) + (-1)^k d(g_{k-1}(n))$, $k \geqslant 2$, then $\{ g_k(n) \}$ must cycle. Of course, these questions can also be asked about many other functions besides $d(n)$.

Consider the k consecutive values $\phi(u+1)$, $\phi(u+2), \ldots, \phi(u+k)$ where $k \leqslant u + k < n$. If we order these k numbers by size then it was proved by Erdös (see [Er (36) a]) that for k small, all possible $k!$ permutations occur and, in fact, every permutation has a density. The same result also holds for $\sigma(m)$, $d(m)$, $v(m)$ and in fact for all decent additive or multiplicative functions. For $k < c_1 \log\log\log n$, all permutations occur but this is not true for $k > c_2 \log\log\log n$. It seems likely that with a little effort one could prove that this holds for $k = c \log\log\log n + o(\log\log\log n)$ (or perhaps even with an error term of $O(1)$). What is the permutation which first fails to appear? Is it $\phi(u+1) > \phi(u+2) > \ldots > \phi(u+k)$? Is it true that the "natural" order, i.e., the order of $\phi(1)$, $\phi(2), \ldots, \phi(k)$ is the most likely.

Denote by $q(x)$ the number of $n \leqslant x$ for which $\phi(m) = n$ is solvable. The fact that $q(x) = o(x)$ was first proved by Pillai [Pil (29)] (also see [Er (35) c]). The sharpest results currently known (due to Erdös and Hall [Er-Ha (76)]) are:

$$\frac{x}{\log x} \exp\left(c(\log\log\log x)^2\right) < q(x) < \frac{x}{\log x} \exp\left(c\sqrt{\log\log x}\right)$$

Does $q(2x)/q(x) \to 2$? An asymptotic formula for $q(x)$ may not exist.

Let $q'(x)$ denote the number of distinct integers in the set

$$\{ \phi(m) : m \leqslant x \}.$$

Of course, $q'(x) \leqslant q(x)$. Does $\lim_x \frac{q(x)}{q'(x)}$ exist? Does it exceed 1?

It was shown [Er (73)] several years ago that the density of integers not of the form $\sigma(n) - n$ is positive. It is very annoying that we cannot show that $\sigma(m) = \phi(n)$ has infinitely many solutions, especially since $\sigma(m) = T$ and $\phi(n) = T$ are both likely to be solvable when T has many prime factors.

It was asked by Erdös and Sierpiński whether there are infinitely many integers not of the form $n - \phi(n)$. If the Goldbach conjecture is valid then every odd number is of this form. What is the situation for even numbers?

With $d(n)$ denoting the number of divisors of n, it can be shown that the set of limit points of $\dfrac{d((n+1)!)}{d(n!)}$ contains $\{1 + 1/k,\ k = 1, 2, \ldots\}$. This also belongs to the set of limit points of the quantities $\dfrac{d((n+2)!)}{d(n!)}$. However, we cannot show that there aren't additional limit points as well. It is easy to show that

$$\frac{d((n+[\sqrt{n}])!)}{d(n!)} \to \infty$$

and, in fact, the term $\sqrt{n}$ can be replaced by $n^{\frac{1}{2}-\varepsilon}$ for a suitable (small) $\varepsilon > 0$. No doubt it is true that

$$d((n+[(\log n)^{\alpha}])!)/d(n!) \to \infty$$

for large α. Probably

$$d((n+[\log n])!)/d(n!)$$

is everywhere dense in $(1, \infty)$ but of course we cannot prove this. More generally, is it true that if $t_1 \leqslant t_2 \leqslant \ldots,\ t_n \to \infty$ and $t_n \leqslant \log n$ then

$$d((n+t_n)!)/d(n!)$$

is everywhere dense?

The following problems were recently raised by Hofstadter [Hofs (77)].

(i) Define $f(n)$ as follows:

$$f(1) = f(2) = 1,$$
$$f(n) = f(n - f(n-1)) + f(n - f(n-2)),\ n > 2.$$

Does $f(n)$ miss infinitely many integers? What is its behavior in general?

(ii) Define a sequence A of integers $a_1, a_2, \ldots$ by starting with $a_1 = 1$, $a_2 = 2$ and thereafter choosing a_k to be the least integer exceeding a_{k-1} which can be represented as the sum of at least two consecutive terms of the sequence. Thus, A begins 1, 2, 3, 5, 6, 8, 10, 11, ... What is the asymptotic behavior of A?

(iii) Define a sequence B of integers $b_1, b_2, \ldots$ as follows. Begin by taking $b_1 = 1$, $b_2 = 2$. In general, if $b_1, \ldots, b_n$ have been defined, form all possible expressions $b_i b_j - 1$, $i \neq j$, and append these to the sequence. B starts with 2, 3, 5, 9, 14, 17, 26, 27, 33, 41, ... Is it true that B has asymptotic density 1?

Is there a sequence $1 \leqslant d_1 < d_2 < \ldots$ with density one so that all the products $\prod_{i=u}^{v} d_i$ are distinct?

Let $a_1 < a_2 < \ldots < a_k \leqslant x$ be a sequence of integers so that the products $a_i a_j$ are all distinct and let $f(x)$ denote the maximum value of k for which this is possible. It has been shown by Erdös [Er (69) b] that there are positive constants c_1, c_2 such that

$$(*) \quad \pi(x) + c_1 x^{3/4}/(\log x)^{3/2} < f(x) < \pi(x) + c_2 x^{3/4}/(\log x)^{3/2}.$$

It is certain that there is a c for which

$$f(x) = \pi(x) + (c + o(1))\, x^{3/4}/(\log x)^{3/2}$$

but this has never been proved.

There are numerous problems of this type dealt with in the literature, so we do not pursue these any further except to mention a surprising development due to R. Alexander which recently occurred. It has often been observed that many extremal problems in number theory can be formulated just as well for reals instead of only integers. However, these more general formulations have only rarely been successfully attacked. For example, suppose $F(x)$ is defined to be the largest integer m for which there exists a sequence of reals $1 \leqslant \alpha_1 < \alpha_2 < \ldots \alpha_m \leqslant x$ so that for all choices of indices i, j, r, s,

$$|\alpha_i \alpha_j - \alpha_r \alpha_s| \geqslant 1.$$

It was suspected for a while that $F(x)$ would also satisfy (*). However, no one was even able to prove that $F(x) = o(x)$. The reason for this lack of success is now apparent from Alexander's proof that $F(x) > x/8e$. Alexander's ingenious construction uses perfect difference sets and is described in [Er (xx) a*].

Can we find $\dfrac{cx}{\log x}$ integers $a_1 < a_2 < \ldots < x$ so that every $m < x$ can be written as $m = a_i + 2^j$ for some i and j ? Ruzsa [Ru (72)] has shown that $\dfrac{cx \log \log x}{\log x}$ such integers can be found.

Is it true that for every n and d there is a k for which

$$p_{n+1} + \dots + p_{n+k} \equiv 0 \pmod{d},$$

where p_r denotes the r^{th} prime?

We can show (assuming the prime k-tuple conjecture) that there is a set $A = \{ a_1 < a_2 < \dots \}$ so that $\limsup_x \frac{A(x)}{\pi(x)} \to 1$ and for infinitely many n, all the integers $n - a_i$, $0 < a_i < n$, are primes. Does this remain true if $\liminf_x \frac{A(x)}{\pi(x)} > 0$ is assumed? It is known [Coh-Se (75)] that for infinitely many n, $n + 2^k$ is always composite and that infinitely many odd integers are not of the form $p + 2^k$. All such proofs which we know of work because there is already a finite set of primes which force these numbers to be composite. Is it possible to prove theorems of the following type: If $a_1 < a_2 < \dots$ tends to infinity rapidly enough and does not cover all residue classes (mod p) for any prime p then for some n, $n + a_i$ is prime for all i? In the other direction—if the a_k do not increase too rapidly then is it true for some n, $n + a_i$ represents all (or almost all) large numbers provided no covering congruence intervenes.

Suppose for a fixed integer n we define a sequence $a_1, a_2, \dots, a_t$, by letting $a_1 = 1$ and for $k \geqslant 2$, defining a_k to be the least integer exceeding a_{k-1} for which all prime factors of $n - a_k > 0$ are greater than a_k. Is it true that for n sufficiently large, not all the quantities $n - a_k$ can be prime? Preliminary calculations made by Selfridge indicate that this is the case but no proof is in sight.

The following very nice problem is due to Ostmann. Are there two infinite sets A and B so that the sum $A + B$ differs from the set of primes by only finitely many elements? Straus modified the problem by asking how dense $A + B$ can be if we assume all the elements of $A + B$ are just pairwise relatively prime. A related old problem: If S has positive lower density, do there always exist infinite sets A and B such that $A + B \subseteq S$?

For integers n and t, define $g(n, t)$ by

$$g(n, t) = \max_{a_i} G(a_1, \dots, a_n)$$

where $0 < a_1 < \dots < a_n \leqslant t$, $\gcd(a_1, \dots, a_n) = 1$ and $G(a_1, \dots, a_n)$ denotes the greatest integer which cannot be expressed as $\sum_{k=1}^{n} x_k a_k$ for any choice of nonnegative integers x_k. The function G was introduced by Frobenius and has been the subject of several dozen papers during the past 10-20 years (see [Er-Gr (72) a], [Bra-Sh (62)], [Jo (60)], [By (74)], [By (75)], [Selm (77)], [Selm-Be (78)], [Rö (78)]). A recent result of the authors [Er-Gr (72) a] proved

$$G(a_1, \ldots, a_n) \leqslant 2a_{n-1} \left[\frac{a_n}{n}\right] - a_n .$$

It follows from this that

$$g(n, t) < 2t^2/n .$$

On the other hand, it is not hard to construct examples showing that

$$g(n, t) \geqslant \frac{t^2}{n-1} - 5t \text{ for } n \geqslant 2 .$$

It is known [Bra (42)] that

$$g(2, t) = (t-1)(t-2) - 1$$

and [Lew (72)]

$$g(3, t) = \left[\frac{(t-2)^2}{2}\right] - 1 .$$

Is it true that

$$g(n, t) \sim \frac{t^2}{n-1} ?$$

Selmer [Selm (77)] very recently has shown under the additional requirement $a_1 \geqslant n$ that

$$G(a_1, \ldots, a_n) \leqslant 2a_{n-1} \left[\frac{a_1}{n}\right] - a_1 .$$

For what choice of k positive integers $a_1 < a_2 < \ldots < a_k \leqslant n$ is the number of integers not of the form $\sum_i c_i a_i$ maximal, where the c_i range over all nonnegative integers? Is the choice $a_i = n - i$ optimal for this?

A related question: For $n \neq p^\alpha$, what is the largest integer not of the form $\sum_{i=1}^{n-1} c_i \binom{n}{i}$ where the c_i are nonnegative integers?

The function $A(k, m)$ was introduced by D. H. and Emma Lehmer [Leh-Leh (62)] some 15 years ago as follows. For a sufficiently large prime p, let $r = r(k, m, p)$ denote the least positive integer such that

$$r, r+1, \ldots, r+m-1$$

are all k^{th} power residues modulo p. Define

$$\Lambda(k, m) = \limsup_{p \to \infty} r(k, m, p).$$

It is known that

$$\Lambda(2,2) = 9, \ \Lambda(3,2) = 77, \ \Lambda(4,2) = 1224,$$
$$\Lambda(5,2) = 7888, \ \Lambda(6,2) = 202124, \ \Lambda(7,2) = 1649375$$
$$\Lambda(3,3) = 23532,$$
$$\Lambda(k,3) = \infty \text{ for all even } k,$$

and

$$\Lambda(k,4) = \infty \text{ for all } k$$

(see [Du (65)], [Mil (65)], [Bri-Leh-Leh (64)], [Leh-Leh-Mi (63)], [Leh + 3 (62)], [Gr (64) c]. Is it true that $\Lambda(k, 2) < \infty$ for all k and $\Lambda(k, 3) < \infty$ for all odd k? If so what are estimates for their values?

Consider a sequence $1 \leqslant a_1 < a_2 < \ldots < a_k \leqslant x$ and look at the partial products $a_1, a_1 a_2, \ldots, a_1 a_2 \ldots a_k$. How many of these products (as a function of x) can be squares? It is trivially $o(x)$ but probably there can be many more than $x^{1/2}$. Perhaps for any $\varepsilon > 0$ there can actually be more than $x^{1-\varepsilon}$.

How large can $A = \{a_1, \ldots, a_k\} \subseteq [1, n]$ be so that no sum $a_i + a_j$ is a square? The integers in $[1, n]$ which are $\equiv 1 \pmod 3$ show that k can be as large as $n/3$. However, k can actually be significantly larger than this (see *Added in proof* p. 107).

If we form a graph G with positive integers as its vertices and edges $\{i, j\}$ if $i + j$ is a square then Erdös and D. Silverman asked: Is the chromatic number of G equal to $\aleph_0$? What if $i + j$ is required to be a k^{th} power?

Let $a_1 < a_2 < \ldots$ be an infinite sequence of integers and denote by $A(x)$ the number of indices i for which $\operatorname{lcm}(a_i, a_{i+1}) \leqslant x$. It seems likely that $A(x) = O(x^{1/2})$. It is easy to give a sequence with $\limsup \dfrac{A(x)}{x^{1/2}} = c$. How large can $\liminf \dfrac{A(x)}{x^{1/2}}$ be (see [Er-Sz (xx)a])?

A related old problem of Erdös asks for the largest value of k for which there exists a sequence $a_1 < \ldots < a_k$ so that $\operatorname{lcm}(a_i, a_j) \leqslant x$ for all i and j. It is conjectured that this maximum value is attained by choosing the sequence consisting of the integers in $[1, \sqrt{n/2}]$ together with the even integers in $[\sqrt{n/2}, \sqrt{2n}]$.

Is it true that if $a_1 < a_2 < \dots$ is a sequence of integers satisfying

$$\frac{1}{\log \log x} \sum_{a_i < x} \frac{1}{a_i} \to \infty$$

then

$$\left(\sum_{a_i < x} \frac{1}{a_i} \right)^{-2} \sum_{1 < a_i < a_j \leq x} \frac{1}{\operatorname{lcm}(a_i, a_j)} \to \infty \text{ ?}$$

Let m and n be positive integers and consider the two sets $\left\{ k(m-k) : 1 \leqslant k \leqslant \frac{m}{2} \right\}$ and $\left\{ l(n-l) : 1 \leqslant l \leqslant \frac{n}{2} \right\}$. Can one estimate the number of integers common to both? Is this number unbounded? It should certainly be less than $(mn)^{\varepsilon}$ for every $\varepsilon > 0$ if mn is sufficiently large.

Let $a_1 < a_2 < \dots$ be an infinite sequence of integers and let $d_A(n)$ denote the number of a_i which are divisors of n. Erdös and Sárközy [Er-Sá (xx)] proved

$$\limsup_{n \to \infty} \max_{n < x} \frac{d_A(n)}{\sum_{a_i < x} \frac{1}{a_i}} = \infty .$$

The proof is surprisingly tricky. Probably it is true that

$$\limsup_{n \to \infty} \max d_A(n) \left(\sum_{a_i < x} \frac{1}{a_i} \right)^{-k} = \infty$$

for every k but we cannot prove this.

Let $x_1, x_2, \dots, x_n$ be n distinct integers. We conjecture that the total number of integers of the form $x_i + x_j$ and $x_i x_j$, $1 \leqslant i < j \leqslant n$, is greater than $n^{2-\varepsilon}$. Szemerédi [Sz $(\pm)$] has very recently proved that the number exceeds n^{1+c} for some $c > 0$.

Erdös and Szemerédi observed that it follows from a theorem of Freĭman [Fre (73)] that this number must grow faster than cn for any c; the first n integers show that it is bounded above by $n^2/(\log n)^{\alpha}$ for some $\alpha > 0$. In fact, it has recently been proved [Er-Sz (xx) b] that it is bounded above by $n^2/e^{\log n/\log \log n}$. Perhaps this is essentially the right order of magnitude.

The same question can be asked for all 2^n sums $\sum_{i=1}^{n} \varepsilon_i x_i$ and products $\prod_{i=1}^{n} x_i^{\varepsilon_i}$, $\varepsilon_i = 0$ or 1. In this case we expect to have more than n^c integers for any c provided $n > n(c)$. Examples can be constructed which only generate $n^{c \log n}$ sums and products.

Is it true that for every $c > 1/2$, if p is a sufficiently large prime then the interval $(u, u+p^c)$ contains two integers a and b satisfying $ab \equiv 1 \pmod p$? A theorem of Heilbronn [He (∞)] guarantees this for c sufficiently close to 1.

Denote by ε_n the density of integers having a divisor in $(n, 2n)$. It was shown long ago [Er (35) a] that $\varepsilon_n < (\log n)^{-\delta}$ and Tenenbaum [Ten (75/76)] has more recently shown that $\varepsilon_n = (\log n)^{-(1+o(1))\alpha}$ where $\alpha = 1 - (1+\log\log 2)/\log 2 = 0.08607...$; however no asymptotic formula for ε_n is currently available. If ε_n' denotes the density of integers having exactly one divisor in $(n, 2n)$, is it true that $\varepsilon_n'/\varepsilon_n \to 0$? Is there a β so that for every $x > n$ there is an $m \in (x, x+(\log n)^\beta)$ which has a divisor in $(n, 2n)$?

An old conjecture of Erdös asserts: Almost all integers n have two divisors d_1 and d_2 with $d_1 < d_2 < 2d_1$. A stronger form of this conjecture is the following: Let $\tau(n)$ denote the number of divisors of n and let $\tau^*(n)$ denote the number of integers k for which n has a divisor d with $2^k \leqslant d < 2^{k+1}$. Is it true that for all $\varepsilon > 0$, $\tau^*(n) < \varepsilon\tau(n)$ for almost all n? At present there is no good inequality known for $\sum_{n=1}^{x} \tau^*(n)$. A very recent problem of Erdös and Hall asks to show that $r(n)$, the number of pairs of divisors d_1, d_2 of n satisfying $d_1 < d_2 < 2d_1$, satisfies for every $\varepsilon > 0$, $r(n) < \varepsilon\tau(n)$ for almost all n.

How large must $y = y(\varepsilon, n)$ be so that the number of integers in $(x, x+y)$ having a divisor in $(n, 2n)$ is less than εy?

Let n_k denote the smallest integer for which $\prod_{i=1}^{k} (n_k - i)$ has no prime factor in $(k, 2k)$. We can prove $n_k > k^{1+c}$ but no doubt much more is true.

Let $1 = a_1 < a_2 < ... < a_{\phi(n)} = n-1$ be the integers relatively prime to n and let $F(n)$ denote $\max_k (a_{k+1} - a_k)$. Erdös [Er (62) b] has shown that almost all n satisfy

$$F(n) = \big(1+o(1)\big)\frac{n\log\log n}{\phi(n)}.$$

Of course, $F(n)$ can be much larger than this for some n. It is true that if $G(n)/F(n) \to \infty$ then for almost all n every interval of length $G(n)$ contains $\big(1+o(1)\big)\, G(n)\dfrac{n}{\phi(n)}$ a_i's. An old conjecture of Erdös asserts:

$$\sum_{k=1}^{\phi(n)-1} (a_{k+1} - a_k)^2 < c\,\frac{n^2}{\phi(n)}$$

for an absolute constant c. It is surprising this is still open. Hooley [Hoo (62)], [Hoo (65) a], [Hoo (65) b] has proved somewhat weaker results. It shouldn't be difficult to prove

$$\sum_{n=1}^{x}\left(\sum_{k=1}^{\phi(n)-1}(a_{k+1}-a_k)^2\right) < cx^2 .$$

It is known [Er (37)] that the density of integers n with $v(n) > \log\log n$ is $1/2$ (where $v(n)$ denotes the number of distinct prime factors of n). It is easy to see by the Chinese remainder theorem that there are at least $t = (1+o(1))\dfrac{\log x}{(\log\log x)^2}$ consecutive integers $n+i$, $1 \leqslant i \leqslant t$, with $v(n+i) > \log\log x$. However, we have no upper bound for this, i.e., as far as we know $\dfrac{\log x}{(\log\log x)^2}$ could be $(\log x)^k$ or even more.

Recently, Pomerance [Pom (xx) b] disproved a conjecture of Erdös and Straus by showing that for infinitely many n and every $i < n$,

$$(\dagger) \qquad p_n^2 > p_{n+i}p_{n-i}$$

where p_k denotes the k^{th} prime. In fact, he proves this holds for any increasing sequence a_n with $a_n^{1/n} \to 1$. No doubt the density of n satisfying (†) is zero but this has not yet been proved. Pomerance also conjectured

$$\limsup_n \frac{1}{p_n}\left(p_n^2 - \max_i p_{n+i}p_{n-i}\right) > 0\,;$$

he can only prove

$$\limsup_n \frac{1}{(\log n)^2}\left(p_n^2 - \max_n p_{n+i}p_{n-i}\right) \geqslant 1.$$

If $f(n)$ denotes $\min_i (p_{n+i}+p_{n-i})$, is it true that

$$\limsup_n \left(f(n) - 2p_n\right) = \infty\ ?$$

Pomerance has proved that the lim sup is at least 2. He has also recently proved the nice (but unrelated) result that if the counting function $A(x)$ of a set $A \subseteq \mathbf{Z}^+$ satisfies $A(x) = o(x)$ then $A(n)$ divides n for infinitely many n. In particular, this implies that $\pi(n)$ divides n infinitely often (see [Mz (77)]).

Let $q_1 < q_2 < \dots$ be a sequence of primes satisfying

$$q_{n+1} - q_n \geqslant q_n - q_{n-1}\,.$$

In [Ric (76)], Richter proves that

$$\liminf_n \frac{q_n}{n^2} > 0 .$$

Is it true that the limit is actually infinite?

Let s_n denote the smallest prime $\equiv 1 \pmod n$ and let m_n denote the smallest integer with $\phi(m_n) \equiv 0 \pmod n$. Is it true that for almost all n, $s_n > m_n$? Does $s_n/m_n \to \infty$ for almost all n? Are there infinitely many primes p such that $p - 1$ is the only n for which $m_n = p$?

Let $g(n, k)$ be the smallest prime which does not divide $\prod_{i=1}^{k} (n+i)$. Is it true that infinitely often

$$g(n, \log n) > (2+\varepsilon) \log n \ ?$$

Denote by A_n the least common multiple of the integers $\{1, 2, \ldots, n\}$ and let p_k denote the k^{th} prime. Almost certainly

$$A_{p_{k+1}-1} < p_k A_{p_k}$$

must hold for every k but the proof of this is certainly beyond our ability, in fact, for two reasons (at least). First of all there could be many squares of primes q^2 with $p_k < q^2 < p_{k+1}$. The proof that one could not have two such q would follow from $p_{k+1} - p_k < p_k^{1/2}$. The small primes also cause intractable trouble. In fact, if A_n' denotes the least common multiple of the numbers $p^\alpha \leqslant n$ with $\alpha \geqslant 2$ then

$$A'_{n+[\sqrt{n}]}/A'_n < n$$

seems extremely unlikely to us.

Given u, let $v = f(u)$ be the largest integer such that no $m \in (u, v)$ is composed entirely of primes dividing uv. Estimate $f(u)$.

The following question arose in work of Eggleton, Erdös and Selfridge [Eg-Er-Se (xx)]. Define $a_0, a_1, a_2, \ldots$ by: $a_0 = n$, $a_1 = 1$, a_k is the least integer exceeding a_{k-1} for which $(n-a_k, n-a_i) = 1$, $1 \leqslant i < k$. Set

$$g(n) = \sum_i \frac{1}{a_i} = \sum_1 + \sum_2$$

where in $\sum_1$, $p(n-a_j) > a_j$ (where $p(t)$ denotes the least prime factor of t). Does $g(n) \to \infty$? Does $\sum_1 \to \infty$? Does $\sum_2 \to \infty$?

Set $n + i = a_i b_i$, $1 \leqslant i \leqslant t$, where all prime factors of a_i are less than t and all prime factors of b_i are greater than or equal to t. Denote by $f(n, t)$ the number of distinct a_i's. Is there an $\varepsilon > 0$ so that

$$\min_t f(n, t)/t > \varepsilon \,?$$

We can only show

$$f(n, t) > \frac{ct}{\log t}.$$

With $p(n)$ denoting the least prime factor of n, it is easy to see that

$$\sum_{\substack{n<x \\ n \neq \text{prime}}} \frac{p(n)}{n} = (c+o(1)) \frac{x^{1/2}}{(\log x)^2}.$$

Is it true that

$$\sum \frac{p(n)}{n} > c'.$$

where n ranges over all integers in $[x, x+cx^{1/2} (\log x)^2]$?

Given c, is it true that for $n > n_0(c)$ there is always a composite number $m > n + c$ for which $m - p(m) < n$?

With p_k denoting the k^{th} prime, let $d_k = p_{k+1} - p_k$. No doubt for r consecutive d_i's, all possible orderings occur (asymptotically with equal probability?). However, we cannot even exclude the possibility that from a certain point on,

$$d_m > d_{m+1}, \; d_{m+2} > d_{m+1}.$$

The sets of integers n for which $\phi(n+1) > \phi(n)$, $\sigma(n+1) > \sigma(n)$ and $d(n+1) > d(n)$ all have density 1/2. However, the problem of whether the density of $\{ n : P(n+1) > P(n) \}$ is 1/2 (where $P(n)$ is the largest prime factor of n) seems very hard (see [Er-Pom (78)]).

Let $a_1 < a_2 < \ldots$ satisfy $\dfrac{a_{k+1}}{a_k} \geqslant c > 1$ for all k. It has very recently been shown that there is an irrational α so that $\{ a_k\alpha - [a_k\alpha] : k = 1, 2, \ldots \}$ is not everywhere dense (settling an old conjecture of Erdös). In fact every interval contains c (the cardinality of the continuum) such α's. A theorem of Erdös and Taylor [Er-Ta (57)] states that the set of α's for which $\{ a_k\alpha - [a_k\alpha] : k = 1, 2, \ldots \}$ is not uniformly distributed has Hausdorff dimension 1.

For the real number x, let $\| x \|$ denote the distance from x to the nearest integer. For points P and Q in the plane, we denote the (Euclidean) distance between P and Q by $d(P, Q)$. Finally, for fixed $X > 0$ and $\delta \in (0, 1/2)$, let $N(X, \delta)$ denote the maximum number of points $P_1, P_2, \ldots, P_n$ which can be chosen in a circle of radius X so that

$$\| d(P_i, P_j) \| \geqslant \delta \quad \text{for} \quad 1 \leqslant i < j \leqslant n.$$

Erdös conjectured that for any $\delta \in (0, 1/2)$,

$$N(X, \delta) = o(X),$$

and, on the other hand, there is a $\delta_0 > 0$ so that

$$\lim_{x \to \infty} N(X, \delta_0) = \infty .$$

The first conjecture was proved by Sárközy [Sár (xx) a] who showed

$$N(X, \delta) \leqslant \frac{4 \times 10^4}{\delta^3} \frac{X}{\log \log X}$$

for X sufficiently large. The second conjecture was proved by Graham [Gr (∞)] who showed

$$N(X, 1/10) > \frac{1}{10} \log X .$$

This was substantially improved by Sárközy [Sár (xx) b] who showed that for an absolute constant c,

$$N(X, 1/10) > X^c .$$

In fact, Sárközy shows that for all $\varepsilon > 0$, if $\delta \leqslant \delta(\varepsilon)$ then

$$N(X, \delta) > X^{1/2 - \varepsilon}$$

for X sufficiently large. There is still a fairly wide gap between the upper and lower bounds. Is it true that for any $\varepsilon > 0$,

$$N(X, \delta) < X^{1/2 + \varepsilon}$$

for X sufficiently large? Unfortunately, we do not even see how to show

$$N(X, \delta) < X^{1 - \varepsilon}$$

for a positive ε.

A problem on sieves: Can one split the primes less than n into two classes $\{ q_i \}$, $\{ q_i' \}$ so that for suitable choices of a_i and a_i', every integer x less than n satisfies $x \equiv a_i \bmod q_i$ and $x \equiv a_i' \bmod q_i'$?

For a given n, let $1 < d_1 < d_2 < \ldots$ be the divisors of n and consider the sums $d_1, d_1 + d_2, d_1 + d_2 + d_3, \ldots$. How many new sums do we get from n, i.e., sums not occurring for $m < n$? When does N first occur as a sum? In particular, if $f(N)$ denotes the least value of n for which N occurs, is it true that $f(N) = o(N)$? (or perhaps just for almost all N?)

Suppose $n = d_1 + \ldots + d_k$ where the d_i are distinct proper divisors of n but this is not true for any proper divisor of n. Must the sum of the reciprocals of all such n converge? Similarly, the same question can be asked for those n which do not have distinct sums of sets of divisors (but any proper divisor of n does).

An integer n has been called *weird* [Ben-Er (74)] if $\frac{\sigma(n)}{n} \geqslant 2$ and $n \neq d_1 + \ldots + d_k$ where the d_i are distinct proper divisors of n. Are there any odd weird numbers? Are there infinitely many *primitive* weird numbers, i.e., so that no proper divisor of n is weird?

The following two problems are due to Ulam. Starting with a given set of primes $Q = \{ q_1, \ldots, q_m \}$, form the set Q' by adjoining to Q all primes formed by adding any three distinct elements of Q. Now repeat this operation on Q', etc. Will the sizes of the generated sets become unbounded provided Q is suitably chosen? What about $Q = \{ 3, 5, 7, 11 \}$?

Starting with a set of primes $Q = \{ q_1, \ldots, q_m \}$ form the sequence $Q^* = (q_1, q_2, \ldots)$ by letting q_{k+1} be the smallest prime of the form $q_n + q_i - 1$, $1 \leqslant i < n$, for $n > m$. For example, if $Q = \{ 3, 5 \}$, $Q^* = \{ 3, 5, 7, 11, 13, 17, \ldots \}$. Is there a choice of Q so that Q^* is infinite? What about $Q = \{ 3, 5 \}$?

Segal [Seg (77)] has recently formulated the following problem. Is there a permutation $a_1, a_2, \ldots$ of the positive integers so that $a_k + a_{k+1}$ is always prime? In particular, C. Watts [Wat (77)] asked whether the "greedy" algorithm always generates such a permutation. In other words, if we define

$$g_1 = 1, \; g_{n+1} = \min \{ x : g_n + x \text{ is prime and } x \neq g_i, i \leqslant n \}$$

then do all positive integers occur as g_k's? Do all primes occur as a sum? Odlyzko [Odl (∞)] has constructed a permutation which settles the first problem, i.e., so that $a_k + a_{k+1}$ is always prime. Whether the greedy algorithm also does this seems very difficult to decide. It has been shown that all integers up to 9990 occur as $g_{k's}$. Segal has also asked whether this can be done for the set $\{ 1, 2, \ldots, n \}$. It seems likely that it can but this is currently not known.

Form the infinite sequence $b_1, b_2, \ldots$ by setting $b_1 = 1$ and defining b_{n+1} to be the least integer which is not an interval sum of $b_1, \ldots, b_n$ (i.e., $b_{n+1} \neq \sum_{u \leq i \leq v} b_i$). Thus, the sequence starts

$$1, 2, 4, 5, 8, 10, 14, 15, 16, 21, 22, 23, 25, 26, 28, \ldots .$$

How does the sequence grow? More generally, suppose $a_1 < a_2 < \ldots$ is a sequence so that no a_n is a sum of consecutive a_i's. Must the density of the a_i's be zero? What about the lower density? Can one say more? (See also the related question in section 6).

An old question of Graham [Gr (71)] asks if for any set $\{ a_1, ..., a_t \}$ of nonzero residues modulo a given prime p, there is always a rearrangement $(a_{i_1}, a_{i_2}, ..., a_{i_t})$ so that all the partial sums $\sum_{k=1}^{m} a_{i_k}$ are distinct modulo p?

A recent related result of Erdös and Szemerédi [Er-Sz (76) a] states that if $a_1, a_2, ..., a_p$ are p nonzero residues modulo a prime p such that there is only one value of k for which $a_{i_1} + a_{i_2} + ... + a_{i_k} \equiv 0 \pmod p$ with $i_1 < i_2 < ... < i_k$ then the a_i assume at most two distinct values modulo p. The proof is unexpectedly complicated.

Is it true that if $a_1, ..., a_k$ are distinct residues modulo p then the pair sums $a_i + a_j$, $i \neq j$, represent at least $2k - 3$ distinct residue classes modulo p (or all of $\mathbf{Z}_p$ if $p \leqslant 2k-3$)? It is surprising this old question of Erdös and Heilbronn [Er-He (64)] is still open.

Very recently White [Wh (78)] proved that if $a_1, ..., a_k$ are distinct elements of a (not necessarily Abelian) group and no subset sum of the a_i's is 0 (the identity of the group) then these subset sums represent at least $2k - 1$ distinct elements of the group. Furthermore, this bound is attained only if $k \leqslant 3$ or the a_i generate a dihedral group.

It has been known since prehistoric times that if $a_1, ..., a_n$ are residues modulo n then some sum $a_{i_1} + ... + a_{i_m}$, $i_1 < ... < i_m$, is $0 \pmod n$. This has been generalized to finite semigroups by Gillam, Hall and Williams [Gi-Ha-Wi (75)], where now some sum is required to be an idempotent of the semigroup. For many more problems and results of this type see [Man (65)], [Er-Gi-Zi (61)], [Sz (70)], [Did (75)], [Did-Ma (73)], [Ol (68)], [Ol (69)], [Ol (75)].

Selfridge [Self (76)] conjectures that a maximum set A of distinct residues modulo p having the property that no subset of A sums to $0 \pmod p$ is given by $\{ -2, 1, 3, 4, 5, ..., t \}$ for an appropriate t. For nonprime p the situation seems to be less clear. Devitt and Lam [Dev-La (74)] have determined the maximum values $a(m)$ for all values of the modulus m up to 50, e.g., $a(42) = 9$, $a(43) = 8$, $a(44) = 9$. They ask: Is $a(m)$ almost always nondecreasing? Is $a(m) = [(-1+\sqrt{8m+9})/2]$ infinitely often? For which m is there a set $A \subseteq \mathbf{Z}_m$ with $|A| = a(m)$ such that no element of A is relatively prime to m? For example, $a(12) = 4$ and this is realized by $A = \{ 3, 4, 6, 10 \}$ or $A = \{ 4, 6, 9, 10 \}$.

Suppose $f : \mathbf{Z} \to \mathbf{Z}$ is a polynomial of degree at least 2 and let $S = \{ f(1), f(2), ... \}$. Is it true that there can never exist a direct sum decomposition $\mathbf{Z} = S + T$, i.e., for no set T can every $z \in \mathbf{Z}$ have a unique representation as $s + t$, $s \in S$, $t \in T$ (see [Sek (59)])?

Consider the set S_p of distinct residues of the form $k\,!\ (\mathrm{mod}\ p)$, $1 \leqslant k < p$ where p is prime. Is it true that $|S_p|/p = 1 - \frac{1}{e} + o(1)$?

It is easy to see that $2^n \not\equiv 1\ (\mathrm{mod}\ n)$ for $n > 1$. An old conjecture of Graham asserts that for all $k \neq 1$, there are infinitely many n so that $2^n \equiv k\ (\mathrm{mod}\ n)$. This is known to be true (see [Gr-Leh-Leh (xx)]) if $k = 2^i$, $i \geqslant 1$, and $k = -1$. D. H. and Emma Lehmer [Leh-Leh (71)] have found solutions with $n < 5.10^9$ for all $k \neq 1$ with $|k| \leqslant 100$. In particular, they very recently finally found the (stubborn) smallest (and still only known) value of $n > 1$ for which $2^n \equiv 3\ (\mathrm{mod}\ n)$. It is $n = 4700063497 = 19 \cdot 47 \cdot 5263229$.

The following attractive conjecture is due to D. J. Newman. Let $x_1, x_2, x_3, \ldots$ be real numbers in the closed interval [0, 1]. Is it true that there are infinitely many m and n such that

$$|x_{m+n} - x_m| \leqslant \frac{1}{n\sqrt{5}}\ ?$$

This is known to be false (see *Added in proof* p. 107).

It is surprising that the following problem offers difficulty. For given integers $a_1, \ldots, a_r, b_1, \ldots, b_r$, let T be the transformation which replace thes integer x by the r integers (possibly not all distinct) $a_i x + b_i$, $1 \leqslant i \leqslant r$. Show that if $\sum_i \frac{1}{a_i} > 1$ then for some bound B, it is not possible to start with 1 and apply T repeatedly until all the resulting integers are distinct and greater than B.

Finally, we mention a very unconventional problem. Define the sequence of integers $(a_1, a_2, \ldots)$ by $a_1 = 1$ and

$$a_{n+1} = [\sqrt{2}(a_n + 1/2)]\,, \quad n \geqslant 1\,.$$

Thus, the sequence begins

$$1, 2, 3, 4, 6, 9, 13, 19, 27, 38, \ldots\ .$$

It has been shown [Gr-Po (70)] that if d_n denotes the difference $a_{2n+1} - 2a_{2n-1}$, $n \geqslant 1$, then d_n is just the n^{th} digit in the binary expansion of $\sqrt{2} = 1.01101000\ldots$. It seems clear that there must be similar results for $\sqrt{m}$ and other algebraic numbers but we have no idea what they are.

10. Remarks on an earlier paper

In this final section we will attempt to give an update of some of the problems in the earlier paper "Quelques problèmes de la théorie des nombres" by Erdös [Er (63)*] which appeared in *L'Enseignement Mathématique* in 1963. Some of these questions have already been mentioned in preceding sections; in this case we will refer the reader to the appropriate section.

To begin with, Problem 3 stated: If $1 < a_1 < \ldots < a_k \leqslant x$ is a sequence of integers such that no a_i divides the product of all the others then $k \leqslant \pi(x)$, the number of primes not exceeding x ([Er (43)], [Sco (44)]). We have the following related result. Let $1 < a_1 < \ldots < a_k \leqslant x$ and assume that all the power products $\prod_{i=1}^{k} a_i^{\alpha_i}$, $\alpha_i \geqslant 0$, are distinct. Then $k \leqslant \pi(x)$. The proof follows easily by a counting argument. For further results in this direction, see [Er (70)].

Problem 4 stated that for any set of 16 consecutive integers, one of them is relatively prime to all the others. Furthermore, 16 is best possible since for any $k > 16$, there is a set of k consecutive integers such that no one of them is relatively prime to all the others. The following related questions were raised in [Er-Se (71) a]. Is it true that for each r there exists $k(r)$ so that for any set of at least $k(r)$ consecutive integers, one of them has at least r prime factors in common with the product of all the others? One could also require that it have at least r prime factors in common with at least *one* of the others.

Let us call an integer n *good* if any set of consecutive integers containing n must always contain a number which is relatively prime to all the others (e.g., all primes are good as is 9 and probably all squares of primes). It is known that the lower density of the good numbers is positive but we cannot decide if their asymptotic density exists (see [Er (65) b]). This question is related to the ancient problem, now completely settled [Er-Se (75)], of showing that the product of consecutive integers is never a power (see the discussion in section 8 as well as [Eg-Se (76)], [Ec-Eg (72)], [Er(75)a*] for further problems and results).

Problem 5 asked for the largest value of k so that for some sequence of integers $1 \leqslant a_1 < a_2 < \ldots < a_k \leqslant x$, no a_i divides the product of any two other a_j's. The best estimate known for max k is given by [Er (38)]: $\pi(x) + c_1 x^{2/3}/\log^2 x < \max k < \pi(x) + c_2 x^{2/3}/\log^2 x$.

This should be strengthened to

$$\max k = \pi(x) + c_3 x^{2/3}/\log^2 x + o(x^{2/3}/\log^2 x)$$

but we cannot do this at present. For related results see [Er (69) a].

Problem 6 asked for the largest value of k so that for some sequence of integers $1 \leqslant a_1 < \dots < a_k \leqslant x$, all products $a_i a_j$, $i < j$, are distinct. It has been shown that

$$\pi(x) + c_1 x^{3/4}/\log^{3/2} x < \max k < \pi(x) + c_2 x^{3/4}/\log^{3/2} x$$

(see [Er (38)], [Er (69) a]). No doubt, as we remarked earlier it is actually true that

$$\max k = \pi(x) + c x^{3/4}/\log^{3/2} x + o(x^{3/4}/\log^{3/2} x).$$

In Problem 7, the following related question was raised. Let $1 \leqslant a_1 < \dots < a_k \leqslant x$ be integers such that all subsets of the a's have distinct products. What is the maximum possible value of k? In [Er (66)], Erdös has shown that

$$k < \pi(x) + c x^{1/2}/\log x .$$

From below, Erdös and Posa have proposed the following construction. For $n \geqslant 1$, let $\beta(n)$ denote the least integer t so that there is a set $\{a_1, \dots, a_n\} \subseteq \{1, 2, \dots, t\}$ which has all its subset sums distinct. Thus, $\beta(1) = 1$, $\beta(2) = 2$, $\beta(3) = 4$, $\beta(4) = 7$ and $\beta(5) = 13$ (by taking $\{6, 9, 11, 12, 13\}$). Then for the primes $p \in (x^{1/t+1}, x^{1/t})$, let A contain the numbers p^{a1}, $p^{a2}, \dots, p^{a_n}$, for all $t \geqslant 1$. Thus, A has all subset products distinct and so

$$\max k \geqslant |A| \sim \sum_{n \geq 1} \pi(x^{1/\beta(n)}) \sim \frac{1}{\log x} \sum_{n \geq 1} \beta(n) x^{1/\beta(n)}$$

$$= \frac{1}{\log x}(x + 2x^{1/2} + 4x^{1/4} + 7x^{1/7} + \dots)$$

Probably,

$$\max k = \pi(x) + \pi(x^{1/2}) + o\left(\left(\frac{x}{\log x}\right)^{1/2}\right).$$

In Problem 9, it was asked whether for two sequences

$$1 \leqslant a_1 < \dots < a_x \leqslant n, \quad 1 \leqslant b_1 < \dots < b_y \leqslant n,$$

such that all products $a_i b_j$ are distinct, it is true that

$$xy < cn^2/\log n .$$

This has now been proved by Szemerédi and appears in [Sz (76)]. Probably

$$\max xy = (1 + o(1)) \frac{n^2}{\log n}$$

but this has not yet been proved (see [Er-Sz (76) b]).

Problem 13 deals with the following question: Estimate $\max_{a_i} \sum_i \frac{1}{a_i}$ where the a_i range over all sequences $1 \leqslant a_1 < \ldots < a_k \leqslant n$ such that $\operatorname{lcm}(a_i, a_j) > n$ for all i, j. Further results on this problem have now appeared in [Er-Sá-Sz (66) a], [Er-Sá-Sz (66) b], [Er-Sá-Sz (68) a].

In Problem 15, $f(n)$ was defined to be the least positive integer such that at least one of the integers $n, n+1, \ldots, n+f(n)$ divides the product of the others. It can be shown that for infinitely many n

$$f(n) > \exp\left((\log n)^{1/2-\varepsilon}\right).$$

Further results on this problem have since appeared in [Er-Se (67)], [Er-Se (71) b].

In Problem 16, the conjecture of Erdös that the density of squares in an arithmetic progression must tend to zero with the length of the progression now follows at once from the result of Szemerédi [Sz (75)] (see also section 2) that a sequence containing no k-term arithmetic progression must have density zero. (In fact, this already follows from Szemerédi's earlier result [Sz (69)] for $k = 4$).

Problem 19 considered the following problem. Suppose $a_1, a_2, \ldots$ is an infinite set of integers which contains no infinite subset B such that $b \nmid b'$ for any two distinct elements $b, b' \in B$. Is it true that the set of elements of the form $\prod_i a_i^{n_i}$ where the n_i are arbitrary integers, has the same property? This theory has been expanded greatly in the direction of set theory. The reader is referred to the papers of Laver, Nash-Williams and Kruskal [Krus (72)], [N-W (68)], [Lav (71)] for discussions of this development.

In Problem 20, the following question was considered. Let $1 < \alpha_1 < \alpha_2 < \ldots$ be a sequence of real numbers which satisfy $|k\alpha_i - \alpha_j| \geqslant 1$ for all i, j, k with $i \neq j$. Is it true that

$$\sum_{\alpha_i < n} \frac{1}{\alpha_i} < c \log n / (\log \log n)^{1/2} \; ?$$

Is it true that

$$\sum_i \frac{1}{\alpha_i \log \alpha_i} < \infty \; ?$$

There are now many results known for such sequences (see the review paper [Er-Sá-Sz (68) a]). It was shown by Haight [Hai (xx)] that if the α_i are rationally independent and the α_i have positive upper density then for all $\varepsilon > 0$ the inequality $| k\alpha_i - \alpha_j | < \varepsilon$ is always solvable in integers i, j, k, with $i \neq j$.

In Problems 20-24, a number of questions and results concerning sequences $a_1 < a_2 < \dots$ for which no a_i divides any a_j, $i \neq j$, were stated. In addition to the bibliography on this topic mentioned there (e.g., [Beh (35)], [Er (35) a], [Er (48) b]) one can also consult [Ha-Ro (66) *], [Er (67)] and [Er-Sá-Sz (68) b], where many new results are surveyed.

Problem 25 stated: Let $a_1 < a_2 < \dots$ be a sequence of integers such that for n sufficiently large, the number $f(n)$ of solutions of $n = a_i a_j$ is positive. Prove that lim sup $f(n) = \infty$. This has now appeared in a paper of Erdös [Er (65) a].

Problem 26 dealt with problems of the following type: Show that the sums $a_i + a_j$, $1 \leqslant i \leqslant j \leqslant k$, formed from the integer sequence $a_1 < a_2 < \dots < a_k$ with $k > 3 \cdot 2^{t-2}$, have at least t distinct prime factors [Er-Tu (34)]. Further problems along this line appear in [Er (76) a*].

Problem 28 considered a number of questions concerned with conditions on a sequence of integers $a_1 < a_2 < \dots$ so that all sufficiently large integers can be expressed as sums of subsets of the a_i. This is exactly the subject of section 6 of this paper where the reader can find many problems and results on this topic.

Conway and Guy [Con-Gu (69)] independently settled the question raised in Problem 31 which asked if it is possible to find $n + 2$ integers

$$1 \leqslant a_1 < a_2 < \dots < a_{n+2} \leqslant 2^n$$

so that all sums $\sum_{k=1}^{n+2} \varepsilon_k a_k$, $\varepsilon_k = 0$ or 1, are distinct. The smallest such example they found has $n = 21$. Whether $n + 3$ integers up to 2^n exist with all distinct subset sums is still not known. Erdös currently offers US $500 for a proof (or disproof) that for every k, $n + k$ such integers less than 2^n can always be found for n sufficiently large.

In Problem 31, $f(x)$ was defined to be the maximum number of terms a sequence of integers

$$1 \leqslant a_1 < a_2 < \dots < a_k \leqslant x$$

can have so that all sums $a_i + a_j$ are distinct. It has been known for some time [Er-Tu (41)] that $f(x) \leqslant x^{1/2} + x^{1/4}$. It is still not known if $f(x) = x^{1/2} + O(1)$. (See section 5 and the book [Ha-Ro (66) *] for further related remarks).

Problem 32 dealt with Kelly's [Ke (57)] result that if $a_1, a_2, \ldots$ is a sequence of integers such that all sufficiently large integers can be written as $a_i + a_j$ (with possibly $i=j$) then all sufficiently large integers can in fact be written as a sum of at most 4 *distinct* a_i. It is still not known if 4 can be replaced by 3. Other problems and results on this subject appear in [Er-Gr (xx)] and in section 5 of this paper.

One question raised in Problem 33 was this. Is it possible to have a sequence $B = \{ b_1 < b_2 < \ldots \}$ of integers with $B(x)$ (the number of $b_i \leqslant x$) satisfying $B(x) \leqslant cx/\log x$ so that every sufficiently large integer can be expressed as $2^i + b_j$ for some i, j ? This has now been proved by Ruzsa [Ru (72)]. It is easy to see that c must be as large as log 2. Erdös has conjectured that

$$B(x) > (\log 2 + \varepsilon)\, x/\log x$$

for some $\varepsilon > 0$ independent of x.

Problem 34 discussed the conjecture of Hanani that if $A = \{ a_1 < a_2 < \ldots\}$ and $B = \{ b_1 < b_2 < \ldots \}$ are two sequences such that all sufficiently large integers can be expressed as $a_i + b_j$ then $\limsup_x \dfrac{A(x)\,B(x)}{x} > 1$. This conjecture has since been disproved by Danzer [Dan (64)]. If we assume

$$\limsup_x \frac{A(x)\,B(x)}{x} = 1$$

then how slowly can $A(x)$ and $B(x)$ grow? In Danzer's example, $A(x)^{-1} > (x^2)!$ Can we have $A(x)^{-1}$ and $B(x)^{-1}$ bounded above by c^x? It has been shown by Sárközy and Szemerédi [Sár-Sz (xx)] that

$$\liminf_x \big(A(x)\,B(x) - x\big) = \infty\,.$$

Problem 35 considered a number of questions concerning essential components, i.e., sets B so that for any set A with Schnirelman density $d_A \in (0, 1)$, we always have $d_{A+B} > d_A$ (e.g., see [Stöh-Wir (56)]). Many new results are now known; for details consult [Wir (74)].

A related question concerns the set $S = \{ x^2 : x = 1, 2, \ldots \}$ of squares, which is known to be an essential component. It is known [Plü (57)] that if

$d_A = \frac{1}{t}$ then $d_{S \cup A} \geqslant \frac{1}{5t^{3/4}}$. It may in fact be true that for t large, the truth is $d_{S \cup A} \geqslant \frac{1}{t^{\varepsilon}}$ for any fixed $\varepsilon > 0$.

The following result of Erdös [Er (62) c] (also see [Ben-Er (74)]) is related to Problem 39. If $a_1 < a_2 < \ldots$ is an infinite sequence such that no a_i is a sum of other a_j's then

$$\sum_i \frac{1}{a_i} < 103 .$$

This was later improved by Levine and O'Sullivan [Lev-O'S (77)] who proved

$$\sum_i \frac{1}{a_i} < 5 .$$

Examples show that it is possible to have

$$\sum_i \frac{1}{a_i} > 2 .$$

Problem 40 asked if for each positive integer k, there are k distinct integers $a_1, \ldots, a_k$ such that all sums $a_i + a_j$, $i < j$, are squares. For $k = 5$, J. Lagrange [Lag (70)] has shown the existence of an infinity of solutions. For $k = 6$, he has given a solution and has shown the existence of infinitely many families having 14 of the 15 sums $a_i + a_j$ being squares [Lag (76)], [Nico (77)].

Problem 41 was concerned with pseudoprimes, i.e., numbers n for which $2^n \equiv 2 \pmod{n}$. Very recently, Pomerance dramatically improved Lehmer's [Leh (49)] old bound on $P(x)$, the number of pseudoprimes less then x, which was $P(x) > c \log x$. He shows [Pom (xx)] that $P(x) > \exp\left((\log x)^{5/14}\right)$ for x sufficiently large. Many papers on pseudoprimes have appeared; the reader is referred the recent book [Rotk (72)] and the long paper [Pom-Self-Wag (xx)] for a survey of some of this work.

Problem 42 dealt with covering congruences. For a discussion of many new problems and results in this subject, see section 3 of this paper.

Problem 43 asked for estimates of the maximum number $f(x)$ of congruences $z \equiv a_i \pmod{n_i}$ with $n_1 < n_2 < \ldots < n_k \leqslant x$ such that no integer satisfies two of the congruences. The conjecture that $f(x) = o(x)$ has now been settled by Erdös and Szemerédi [Er-Sz (68)] who showed that for a suitable $c > 0$, $f(x) < x/(\log x)^c$.

Problem 44 stated the following question. What is the largest possible value of $k = k(n)$ for which there exist integers $r_1, r_2, \ldots, r_k$ such that the congruences

$$\sum_{i=1}^{k} \varepsilon_i r_i \equiv 0 \pmod{n}$$

have no solutions for any choice of $\varepsilon_i = 0$ or 1? The conjecture that $k(n) < c\sqrt{n}$ has now been settled by Szemerédi [Sz (70)] and Olson [Ol (75)] (also see [Ol (68)], [Did (75)], [Man (65)] and other references in section 9. The following related conjecture is of some interest and may be quite difficult. Is it true that for every $\varepsilon > 0$, there is an $f(\varepsilon)$ so that if $a_1, \ldots, a_k$, $k = [p^{\varepsilon}]$, are the residues $\frac{1}{i}$ modulo p, $1 \leqslant i \leqslant k$ (where p is prime), then every residue modulo p is the sum of at most $f(\varepsilon)$ a_i's?

In Problem 46, there are of course still no odd perfect numbers known. However, the four largest known perfect numbers are now $(2^{19937}-1)\,2^{19936}$ (see [Tu (71)] $(2^{21701}-1)\,2^{21700}$ [Nic-No (78)], $(2^{23209}-1)\,2^{23208}$ [Nic-No (78)] and $(2^{44497}-1)\,2^{44496}$ [Nel-Slo (79)]. Wirsing can show that for any rational a, the number of $n < x$ such that $\frac{\sigma(n)}{n} = a$ is less than $cx^{c' \log\log\log x / \log\log x}$, independent of a. This follows from arguments he uses in [Wir (59)].

In Problem 47, the conjecture of Catalan on aliquot series was considered. This is that the sequence $\sigma_k(n)$, $k = 1, 2, \ldots$, where $\sigma_1(n) = \sigma(n) - n$, $\sigma_k(n) = \sigma_1(\sigma_{k-1}(n))$, is bounded for any fixed n (and thus, periodic). Extensive computations have recently been carried out by the Lehmers, Guy and Selfridge (see [Gu-Se (75)], [Er (76)], [Gu+3 (74)]), (who all have strong doubts about truth of the conjecture, however).

Problem 48 dealt with simple diophantine equations involving $\phi(n)$, $\sigma(n)$ and n. Lehmer's old (1932) conjecture that $\phi(n) \mid n-1$ implies n is prime [Leh (32)] is still open. A related conjecture of Graham asserts that for any k there are infinitely many n such that $\phi(n)$ divides $n + k$. This is known to be true for infinitely many k. Pomerance [Pom (75)] has shown that the number of composite integers $n < x$ with $\phi(n) \mid n-1$ is less than $cx^{1/2}(\log x)^{3/4}$. Erdös [Er (73)] has succeeded in showing that there are infinitely many n which are not of the form $\sigma(k) - k$. In fact, these n have positive upper density. This is still not known for $k - \phi(k)$. As we mentioned earlier, one of the most annoying problems here is to show that $\sigma(m) = \phi(n)$ has infinitely many solutions. It must be true but at present we have no idea how to prove it. It is still not known if $\phi(n) = \phi(n+1)$ has infinitely many solutions or even if for each $\varepsilon > 0$, of $|\phi(n+1) - \phi(n)| < n^{\varepsilon}$ has infinitely many solutions.

In Problems 49 and 51, various aspects of van der Waerden's theorem on arithmetic progressions were discussed. Quite a lot new has since been discovered including Szemerédi's powerful result that any infinite set of integers having no k-term arithmetic progression must have density zero. This material is covered in section 2 of this paper. The longest arithmetic progression of primes then known was Golubev's 12-term sequence $23143 + 30030k$, $0 \leqslant k \leqslant 11$. As mentioned in section 2, this has been improved by Weintraub [Weint (77)] who found a 17-term arithmetic progression of primes (see also [Kar (69)].

Problem 50 stated: If $f(n)$ is multiplicative function (i.e., $f(a, b) = f(a) f(b)$ for $(a, b) = 1$) taking only values ± 1, must

$$\lim_{n} \frac{1}{n} \sum_{k=1}^{n} f(k)$$

exist? This has been settled completely by Wirsing [Wir (67)] and considerably extended by Halász [Halá (71)].

In Problem 52, it was asked if for some constant c,

$$\sum_{k=0}^{\phi(n)} (a_{k+1} - a_k)^2 < c \frac{n^2}{\phi(n)}$$

where $1 = a_1 < a_2 < \ldots < a_{\phi(n)} = n - 1$ are the integers less than n which are relatively prime to n (and $a_0 = -1$). This is still unsettled although Hooley [Hoo (62)], [Hoo (65) a], [Hoo (65) b] has some partial results.

In Problem 54, it was conjectured that for $n > 105$, $n - 2^k$ cannot be prime for all k such that $2 \leqslant 2^k < n$. This is still not known; however, Vaughan [Va (73)] has some modest upper bounds on the number of such n less than x. The bounds are rather weak but at present they are all we have. In this problem, it was weakly conjectured that if an infinite sequence of integers $a_1 < a_2 < \ldots$, satisfies $a_{k+1} < ca_k$, then there are only finitely many values of n such that $n - a_k$ is prime for all $a_k < n$. There now are reasons for doubting the truth of this conjecture. It may still hold if we assume the a_k do not grow too fast, e.g., $a_k < ck \log k$. However, if we only require that $a_k < ck^2$ then it might well fail (though we will never live to see this decided). The following question is related to this material. Denote by $f(k)$ the largest integer so that from any k integers one can always select $f(k)$ of them which do not form a complete set of residues modulo p

for any prime p. Clearly $f(k)$ is of the order of magnitude $k/\log k$. An asymptotic formula would be desirable and perhaps will not be difficult to obtain. An explicit formula is probably hopeless.

Elliott [Ellio (65)] has considered the following related problem. How many integers $1 \leqslant a_1 < \dots < a_k \leqslant x$ can be chosen so that the a_k's do not form a complete residue system modulo p for any prime p. Elliott shows

$$k < (2+\varepsilon)\, x/\log x\,;$$

an example of Davenport gives

$$k > \big(1 + o(1)\big)\, x/\log x\,.$$

What is the right coefficient here? Suppose instead that the a_k's form a complete set of residues for at most one (or t) primes. Now what can be said about max k?

The following question is related to Problem 56. Denote by $f(p_j)$ the sum $\sum\limits_{p_i < p_j} \dfrac{1}{p_j - p_i}$ where p_k denotes the k^{th} prime. Probably

$$\liminf_{j \to \infty} f(p_j) = 1$$

but we cannot even prove

$$\frac{1}{n} \sum_{j=1}^{n} f(p_j) \to 1\,.$$

In Problem 57, the sum

$$g(n) = \sum \frac{1}{p}\,,$$

summed over all primes $p \leqslant n$ which do not divide $\binom{2n}{n}$, is discussed. It is still not known if $g(n)$ is bounded. Many related results for $g(n)$ are discussed in section 8 (see also [Er + 3 (75)]).

In Problem 59, sequences $1 \leqslant a_1 < a_2 < \dots$ were considered such that all positive integers can be expressed as $a_i a_j$ (see [Wir (57)]). Suppose $A(x) < cx/\log x$ for infinitely many x (where as usual $A(x)$ denotes the number of a_k's $\leqslant x$). Is it true that $A(x) > c'x$ for infinitely many x. Assume instead that the density of integers of the form $a_i a_j$ is positive. Then it can be shown that $A(x) > x^{1/2+\alpha}$ where α depends on the density. This bound is essentially best possible although the dependence of α on the density may be hard to determine exactly.

In Problem 60, it was asked whether the only pairs of integers m and n having the same prime factors so that $m + 1$ and $n + 1$ also have the same prime factors are given by taking $m = 2^k - 2$, $n = 2^k(2^k - 2)$. In [Mak (68)], Makowski finds other solutions.

In Problem 61, it was asked if the sequence $u_1 < u_2 < \dots$ of integers of the form $x^2 + y^2$ satisfies

$$u_{k+1} - u_k = o(u_k^{1/4})\ ?$$

This is still open.

Problem 62 dealt with the diophantine equation of Ko [Ko (40)]:

$$x^x y^y = z^z .$$

It is surprising that all solutions to this equation have not yet been classified (see [Mil (59)]). In particular, are there any odd solutions? The late Claude Anderson (at Berkeley) conjectured that the related equation

$$x^x y^y z^z = w^w$$

has no nontrivial solution (i.e., with $1<x<y<z$). There are still no nontrivial solutions known to $n! = a!\,b!$ (i.e., $a, b \leqslant n-2$) except $10! = 7!\,6!$. For related results, see [Ab-Er-Ha (74)] and section 8 of this paper.

In Problem 64, it was asked if for integers $a > b > c > 0$ with $a + b + c = n$ a sufficiently large fixed integer, all the products abc must be distinct. J. B. Kelly [Ke (64)] showed that is certainly not the case.

In Problem 65, it was asked if there were any integers $k > 24$ for which the equation $x_1 + \dots + x_k = x_1 \dots x_k$ has a unique solution in positive integers (up to order). It is known to have a unique solution when $k = 2, 3, 4, 6$ and 24. In [Star (71)] it was noted that this is also true for $k = 114, 174, 444$ and for no other values of $k < 10^4$. Perhaps 444 is the largest possible value of such a k.

Problems 69-75 were concerned with numerous questions on sums of unit fractions. Very much new material is now known. Much of this is mentioned in section 4 of this paper.

Finally, Problem 76 asked the following. Does there exist for all $\varepsilon > 0$ and all $n > n_0(\varepsilon)$ a sequence $1 < a_1 < \dots < a_k \leqslant n$ with $k > n(1-\varepsilon)$ such that if two subsets of a_k's have equal products then the subsets have equal cardinalities? This has now been solved in the negative by Ruzsa (see [Er-Ru-Sá (73)].)

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Added in proof

(i) (p. 13) Very recently, T. Brown and J. Buhler have shown that for all $\varepsilon > 0$, if $R \subseteq GF(3)^n$ with $|R| > \varepsilon \cdot 3^n$ and n is sufficiently large then R must contain three points which form an *affine* line in $GF(3)^n$. While this is weaker than showing the existence of a (combinatorial) line, it does offer a bit more evidence for the truth of the general density conjecture.

(ii) (p. 87) It was just observed by J. P. Marsias that the sum of any two integers $\equiv 1, 5, 9, 13, 14, 17, 21, 25, 26, 29, 30 \pmod{32}$ is never a square (mod 32). Thus, k can be chosen to be at least $\frac{11n}{32}$. This is best possible for the modular version of the problem since it has even more recently been shown by J. Lagarias, A. M. Odlyzko and J. Shearer that if $S \subseteq \mathbf{Z}_n$ and $S + S$ contains no square of $\mathbf{Z}_n$ then $|S| \leqslant \frac{11n}{32}$.

(iii) (p. 96) It has just been proved by Chung and Graham that if $x_1, x_2, x_3, \ldots \in [0, 1]$ then for any $\varepsilon > 0$, there is some n such that for infinitely many,

$$|x_{m+n} - x_m| < \frac{1}{(\alpha_0 - \varepsilon)\, n}$$

where

$$\alpha_0 = 1 + \sum_{k \geq 1} \frac{1}{F_{2k}} = 2.535 \ldots$$

and F_m denotes the m^{th} Fibonacci number. Furthermore, this is best possible in that α_0 cannot be replaced by any larger constant (which is shown by taking, for example, $x_k = \{\tau k\}$ with $\tau = \frac{-1+\sqrt{5}}{2}$).

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